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# Metodi Matematici della Meccanica Quantistica

## Assignment 6/6

To be handed in on **Wednesday, January 10, 2024, before 10:30** via email (scanned or LaTeX) to sascha.lill@unimi.it or on paper at the beginning of the lecture.

### Problem 1: Multiplication Operators (5+5+5 points)

**a.** Let $f : \mathbb{R}^n \to \mathbb{R}$ measurable. Show that for all $g \in \mathcal{B}(\mathbb{R})$ we have

$$g(T_f) = T_{g \circ f} \ .$$

**b.** Let $A = A^*$, $U$ unitary, $B := UAU^{-1}$. Show that for all $f \in \mathcal{B}(\mathbb{R})$ we have

$$f(B) = Uf(A)U^{-1} \ .$$

**c.** Let $A = -i\partial$ on $L^2(\mathbb{R})$ (the momentum operator). Compute as explicitly as possible, for $\Omega \subset \mathbb{R}$ a Borel set, the expectation value

$$\langle \varphi, \chi_\Omega(A)\varphi \rangle \ , \quad \varphi \in L^2(\mathbb{R}) \ .$$

### Problem 2: Inverse Function (5 points)

Let $A = A^*$, let $f : \sigma(A) \to \mathbb{R}$ a measurable injective function with measurable inverse function $f^{-1}$ (defined on the image of $f$). Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.

### Problem 3: Commuting Operators (5 points)

Let $A, B$ be commuting self-adjoint operators on a Hilbert space $\mathcal{H}$. Show that

$$f(A)g(B) = g(B)f(A) \quad \text{for all } f, g \in \mathcal{B}(\mathbb{R}) \ .$$

### Problem 4: Domain (5 points)

Under the assumptions of Lemma 10.6 of the lecture, show that

$$D(\phi(f)) = \{\varphi \in \mathcal{H} \mid \int |f|^2 \mathrm{d}\mu_\varphi < \infty\} \ .$$

**Problem 5: An Integral Resolvent Representation (2+4+4 points)**

Let $A = A^*$ be an operator on a Hilbert space $\mathcal{H}$, satisfying $A \geq 0$ (recall that this means $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{H}$).

**a.** Show that $\sigma(A) \subset [0, \infty)$.

**b.** Show that there exists a $c \in [0, \infty)$ such that

$$\sqrt{A} = c \int_0^\infty \left(1 - \lambda^2 \left(A + \lambda^2\right)^{-1}\right) \mathrm{d}\lambda \,.$$

**c.** Let us now simplify to the finite-dimensional case $\mathcal{H} = \mathbb{C}^n$. Let $D \geq 0$ be diagonal in the canonical basis, let $v = (1, 1, 1, \ldots, 1)^T \in \mathbb{C}^n$ (also in the canonical basis) and $P_v$ the rank-one orthogonal projection on $v$. Compute $\sqrt{A}$ as explicitly as possible. (*Hint:* Look up the Sherman-Morrison formula and apply it to the resolvent.)