Metodi Matematici della Meccanica Quantistica

Solutions for Assignment 3

Discussed on **Friday, November 10, 2023**.
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Problem 1: Proof of the Weyl criterion (10 points)

We have to show that for $A : D \subset X \to X$, with X being a Banach space, for $\lambda \in \mathbb{C}$, the existence of a **Weyl sequence** $(x_n)_{n\in\mathbb{N}} \subset D$ with $\|x_n\| = 1, \lim_{n\to\infty} \|(A-\lambda)x_n\| = 0$ implies that $\lambda \in \sigma(A)$. That is, $(A-\lambda)$ has no bounded inverse $(A-\lambda)^{-1} : X \to X$.
If for some x_n, we should have $\|(A-\lambda)x_n\| = 0 \Rightarrow (A-\lambda)x_n = 0$, then it is clear that $(A-\lambda)^{-1}$ can never exist, since we would have $(A-\lambda)^{-1}(A-\lambda)x_n = 0$, which is never x_n.
Now, assume that $(A-\lambda)x_n \neq 0 \;\forall n \in \mathbb{N}$ and suppose some bounded inverse $(A-\lambda)^{-1} : X \to X$ of $(A-\lambda)$ would exist. Then, the sequence $y_n := \frac{(A-\lambda)x_n}{\|(A-\lambda)x_n\|}$ is well-defined with $\|y_n\| = 1$ and we have

$$\|(A-\lambda)^{-1}y_n\| = \frac{\|(A-\lambda)^{-1}(A-\lambda)x_n\|}{\|(A-\lambda)x_n\|} = \frac{\|x_n\|}{\|(A-\lambda)x_n\|} \to \infty \,, \tag{1}$$

so the operator $(A-\lambda)^{-1}$ is unbounded, which yields a contradiction. □

Problem 2: Coulomb potential (5+5 points)

a. We need to show that for every $V \in L^2 + L^\infty(\mathbb{R}^3)$ and every $\varepsilon > 0$, we can split $V = V_2^\varepsilon + V_\infty^\varepsilon$ such that $V_\infty^\varepsilon \in L^\infty(\mathbb{R}^3)$ and $V_2^\varepsilon \in L^2(\mathbb{R}^3)$ with $\|V_2^\varepsilon\|_{L^2} < \varepsilon$.
Recall that $V \in L^2 + L^\infty$ means we can split $V = V_2 + V_\infty$ with $V_2 \in L^2, V_\infty \in L^\infty$. We will now "transfer" parts of V_2 into V_∞ to make the L^2-norm small. This can conveniently be done introducing the (measurable) level sets[1]

$$\chi_L := \{x \in \mathbb{R}^3 \;:\; |V_2(x)| \leq L\} \,, \qquad L > 0 \,. \tag{2}$$

We then split

$$V_2 = V_{2,\leq L} + V_{2,>L}, \qquad V_{2,\leq L}(x) := \begin{cases} V_2(x) & \text{if } x \in \chi_L \\ 0 & \text{else} \end{cases} \,, \quad V_{2,>L} := \begin{cases} 0 & \text{if } x \in \chi_L \\ V_2(x) & \text{else} \,. \end{cases} \tag{3}$$

[1]Strictly speaking, since V_2 is only defined up to modifications on a null set, also χ_L is defined up to modifications on a null set. So $\chi_L = [\chi_L]$ is actually an equivalence class, where two representatives, say $\chi_{L,1}$ and $\chi_{L,2}$, are allowed to differ by a null set. However, for any integrable u, we have $\int_{\chi_{L,1}} u = \int_{\chi_{L,2}} u$ since adding or removing a null set to/from the domain does not change the integral. So the value of integrals like $\int_{\chi_L} u$ is unique.

Obviously, $V_{2,\leq L} \in L^\infty(\mathbb{R}^3)$ with $\|V_{2,\leq L}\|_{L^\infty} \leq L$. Further, the set $\mathbb{R}^3 \setminus \chi_L$ converges pointwise to $\emptyset$ as $L \to \infty$, so

$$\int_{\chi_L} |V_{2,>L}(x)|^2 \mathrm{d}x \to 0 \quad \Rightarrow \quad \|V_{2,>L}\|_{L^2} \to 0 \quad \text{as } L \to \infty \,. \tag{4}$$

We can thus find some large enough $L > 0$, such that $\|V_{2,>L}\|_{L^2} < \varepsilon$. Setting $V_2^\varepsilon := V_{2,>L}$ and $V_\infty^\varepsilon := V_\infty + V_{2,\leq L}$ then achieves the desired split. □

b. We need to show that the Coulomb potential $V(x) := \frac{1}{|x|}$, $x \in \mathbb{R}^3 \setminus \{0\}$ is in $L^2 + L^\infty$. As above, we work with the level sets

$$\chi_L := \{x \in \mathbb{R}^3 \ : \ |V(x)| \leq L\} \,, \qquad L > 0 \,. \tag{5}$$

Setting $L = 1$, we split

$$V = V_{\leq 1} + V_{>1}, \qquad V_{\leq 1}(x) := \begin{cases} V(x) & \text{if } x \in \chi_1 \\ 0 & \text{else} \end{cases} \,, \quad V_{>1} := \begin{cases} 0 & \text{if } x \in \chi_1 \\ V(x) & \text{else} \end{cases} \,. \tag{6}$$

Clearly, $V_{\leq 1} \in L^\infty$ with $\|V_{\leq 1}\|_{L^\infty} = 1$. Further, the pole of $x \mapsto |V_{>1}(x)|^2$ at $x = 0$ is of order $2 < 3$, so $V_{>1} \in L^2$. More precisely, using spherical coordinates,

$$\|V_{>1}\|_{L^2}^2 = \int_{\mathbb{R}\setminus\chi_L} |V(x)|^2 \mathrm{d}x = \int_0^1 \frac{1}{r^2} 4\pi r^2 \mathrm{d}r = 4\pi \,. \tag{7}$$

Thus, $V = V_{\leq 1} + V_{>1}$ is a suitable split to show $V \in L^2 + L^\infty$. □

Problem 3: Sobolev inequalities (5+5 points)

a. Our goal is to find the exponent $q = q(n,p)$, for which the Sobolev inequality can hold:

$$\|f\|_{L^q} \leq C_{n,p,q} \|\nabla f\|_{L^p} \,, \tag{8}$$

while using a rescaling argument via $f_\lambda(x) = f(\lambda x), \lambda > 0$. The rescaling argument bases on the fact that (8) has to hold for any f_λ in place of f. That is,

$$\|f_\lambda\|_{L^q} \leq C_{n,p,q} \|\nabla f_\lambda\|_{L^p} \tag{9}$$

has to hold. We evaluate the norm on the left-hand side, using a substitution $y = \lambda x$

$$\|f_\lambda\|_{L^q} = \left(\int_{\mathbb{R}^n} |f(\lambda x)|^q \mathrm{d}x \right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} |f(y)|^q \lambda^{-n} \mathrm{d}y \right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}} \|f\|_{L^q} \,. \tag{10}$$

For $\|\nabla f_\lambda\|_{L^p}$, we make use of the chain rule with $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(x) := \lambda x$, whose Jacobian is $Dg(x) = \lambda \, \mathrm{id}_{n\times n}$:

$$\nabla f_\lambda(x) = \nabla(f \circ g)(x) = Dg(x) \cdot (\nabla f)(g(x)) = \lambda (\nabla f)(\lambda x) \,. \tag{11}$$

Thus, substituting $y = \lambda x$ yields

$$\|\nabla f_\lambda\|_{L^p} = \left(\int_{\mathbb{R}^n} |\lambda(\nabla f)(\lambda x)|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} |\nabla f(y)|^p \lambda^{p-n} \mathrm{d}y\right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}} \|\nabla f\|_{L^p} \ . \tag{12}$$

As (8) is valid, (9) can only hold for all $\lambda > 0$ if

$$\lambda^{1-\frac{n}{p}} = \lambda^{-\frac{n}{q}} \quad \Leftrightarrow \quad 1 - \frac{n}{p} = -\frac{n}{q} \quad \Leftrightarrow \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n} \quad \Leftrightarrow \quad q = \frac{pn}{n-p} \ , \tag{13}$$

which is the condition on q, we were looking for. Note that one commonly calls q the **Sobolev conjugate** of (n, p). □

b. Here, we wish to show that the **Sobolev embedding** $H^m(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ is true for $m > \frac{n}{2}$. This requires proving

$$\|u\|_{L^\infty} \leq C_{m,n} \|u\|_{H^m} \tag{14}$$

for any $u \in H^m(\mathbb{R}^n)$ and a suitable $C_{m,n} > 0$ uniform in u. The Sobolev norm is conveniently expressed[2] using the Fourier transform

$$\hat{u}(k) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) e^{-ikx} \mathrm{d}x \quad \Leftrightarrow \quad u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(k) e^{ikx} \mathrm{d}k \tag{15}$$

and **Plancherel's theorem** $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$:

$$\|u\|_{H^m} = \left(\sum_{|\alpha|<m} \|D^\alpha u\|_{L^2}^2\right)^{\frac{1}{2}} = \left(\sum_{|\alpha|<m} \|k^\alpha \hat{u}\|_{L^2}^2\right)^{\frac{1}{2}} \ . \tag{16}$$

The L^∞-norm is bounded by

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |u(x)| \leq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\hat{u}(x)| \mathrm{d}x = (2\pi)^{-\frac{n}{2}} \|\hat{u}\|_{L^1} \ . \tag{17}$$

Now, observe that for $m > \frac{n}{2}$, the function

$$f(k) := \frac{1}{1 + k_1^m + \ldots + k_n^m} \tag{18}$$

decays at $|k| \to \infty$ faster than $|k|^{-\frac{n}{2}}$, so $f \in L^2(\mathbb{R}^n)$ with $\|f\|_{L^2} =: c_{m,n}$. Thus, we can apply Hölder's inequality as

$$\begin{aligned}\|\hat{u}\|_{L^1} = & \left\|f \frac{1}{f} \hat{u}\right\|_{L^1} \leq \|f\|_{L^2} \left\|\frac{1}{f} \hat{u}\right\|_{L^2} \leq c_{m,n} \left(\|\hat{u}\|_{L^2}^2 + \|k_1^m \hat{u}\|_{L^2}^2 + \ldots + \|k_n^m \hat{u}\|_{L^2}^2\right)^{\frac{1}{2}} \\ \leq & c_{m,n} \|u\|_{H^m} \ .\end{aligned} \tag{19}$$

[2] Here, we use the standard multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n),\ \alpha_j \in \mathbb{N}$, with $|\alpha| = \sum_{j=1}^n \alpha_j,\ D^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n},\ k^\alpha = k_1^{\alpha_1} \cdot \ldots \cdot k_n^{\alpha_n}$.

Combining with (17), this yields the desired Sobolev bound (14).
Note that sometimes, one uses the different convention $\|u\|_{H^m} := \|\langle k\rangle^m \hat{u}\|_{L^2}$ with the **Japanese symbol** $\langle k\rangle := \sqrt{1+|k|^2}$. The proof is then analogous with $f(k) := \langle k\rangle^{-m}$. $\square$

Problem 4: On the Existence Proof for SCUGs (5+5 points)

Let $A : D \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

In the lecture we defined

a. Recall that $A : D \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint, $B_m := im(A + im)^{-1},\ m \in \mathbb{Z},\ A_m := B_m A B_{-m}$ and

$$U_m(t) := e^{-iA_m t} := \sum_{k\in\mathbb{N}} \frac{1}{k!} (-itA_m)^k \ . \tag{20}$$

We have to show that $U_m(t)$ is a strongly continuous unitary group (SCUG) and that $U(t) := \text{s-lim}_{m\to\infty} U_m(t)$ exists. Checking that $U_m(t)$ is a SCUG amounts to verifying the three axioms:

- First, $U_m(t)$ is unitary: It is easy to see that A_m is bounded since $B_{\pm m}$ are bounded and

$$B_m A = B_m(A + im - im) = im - im(A + im)^{-1} im \tag{21}$$

is also bounded. Further, it is easy to check that $B_m^* = B_{-m}$, so A_m is self-adjoint. Now, $U_m(t)^* = \sum_k \frac{1}{k!} (itA_m^*)^k = \sum_k \frac{1}{k!} (itA_m)^k$, which indeed agrees with $U_m(t)^{-1}$, as by the Baker-Campbell-Hausdorff (BCH) formula,

$$U_m(t)U_m(t)^* = e^{-itA_m} e^{itA_m} = e^{-itA_m + itA_m} = e^0 = 1 \ . \tag{22}$$

- $U_m(t+s) = e^{-i(t+s)A_m} = e^{-itA_m} e^{-isA_m} = U_m(t)U_m(s)$ due to BCH.

- Finally, we have strong continuity, as for $\varphi \in \mathcal{H}$:

$$\lim_{t\to 0} \|U_m(t)\varphi - \varphi\| = \lim_{t\to 0} \left\| \sum_{k=1}^{\infty} \frac{1}{k!} (-itA_m)^k \varphi \right\| \leq \lim_{t\to 0} \sum_{k=1}^{\infty} \frac{|t|^k}{k!} \|A_m\|^k \|\varphi\| = 0 \ . \tag{23}$$

Next, we show that $U(t)\varphi := \lim_{m\to\infty} U_m(t)\varphi$ exists for all $\varphi \in D,\ t \in \mathbb{R}$. By completeness of $\mathcal{H}$, this is true if we can show that $(U_m(t)\varphi)_{m\in\mathbb{N}}$ is a Cauchy sequence.

To do so, we use the fundamental theorem of calculus, as well as the fact that A_n, A_m commute with $U_m(t)$ (which can easily be checked by explicit computation):

$$\begin{aligned} U_n(t)\varphi - U_m(t)\varphi &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(U_n(s)U_m(t-s))\varphi \mathrm{d}s = i\int_0^t U_n(s)(A_m - A_n)U_m(t-s)\varphi \mathrm{d}s \\ &= i\int_0^t U_n(s)U_m(t-s)(A_m - A_n)\varphi \mathrm{d}s \; . \end{aligned} \tag{24}$$

Thus, using $\|U_n(t)\| = 1$,

$$\begin{aligned} \|U_n(t)\varphi - U_m(t)\varphi\| &\leq |i| \int_0^t \|U_n(t)\| \|U_m(t-s)\| \|A_m\varphi - A_n\varphi\| \mathrm{d}s \\ &= |t| \|A_m\varphi - A_n\varphi\| \; . \end{aligned} \tag{25}$$

Now, since $(A_n\varphi)_{n\in\mathbb{N}}$ is a Cauchy sequence, following (25), so is $(U_m(t)\varphi)_{m\in\mathbb{N}}$.
The statement can now easily be generalized from $\varphi \in D$ to $\varphi \in \mathcal{H}$: We approximate $\varphi \in \mathcal{H}$ by a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset D$ in order to show that $(U_m(t)\varphi)_{m\in\mathbb{N}}$ is a Cauchy sequence:

$$\|U_m(t)\varphi - U_n(t)\varphi\| \leq \|U_m(t)\varphi - U_m(t)\varphi_k\| + \|U_m(t)\varphi_k - U_n(t)\varphi_k\| + \|U_n(t)\varphi_k - U_n(t)\varphi\| \; . \tag{26}$$

All three terms can be made arbitrarily small by choosing k large enough. Therefore, $(U_m(t)\varphi)_{m\in\mathbb{N}}$ is indeed a Cauchy sequence. □

b. We have to show that $U(t)$ is a SCUG and its generator is A. First, we verify the axioms of a SCUG:

- $U(t)$ is unitary: First, observe that $\forall \psi, \varphi \in \mathcal{H}$,

$$\langle \psi, U(t)\varphi \rangle = \lim_{m\to\infty} \langle \psi, U_m(t)\varphi \rangle = \lim_{m\to\infty} \langle U_m(t)^*\psi, \varphi \rangle \; , \tag{27}$$

so $U(t)^* = \lim_{m\to\infty} U_m(t)^*$. Now, by unitarity of $U_m(t)$ and the fact that the limit of products is the product of limits (if it exists):

$$1 = \lim_{m\to\infty} U_m(t)^* U_m(t) = U(t)^* U(t) \; , \tag{28}$$

so $U(t)^*$ is indeed $U(t)^{-1}$.

- $U(t+s) = \text{s-lim}_{m\to\infty} U_m(t+s) = \text{s-lim}_{m\to\infty} U_m(t)U_m(s) = U(t)U(s)$.

- Strong continuity: For any $m \geq 1$, it holds true that

$$\|U(t)\varphi - \varphi\| \leq \|U(t)\varphi - U_m(t)\varphi\| + \|U_m(t)\varphi - \varphi\| \; . \tag{29}$$

Now, for any given $\varepsilon > 0$, we can achieve $\|U(t)\varphi - U_m(t)\varphi\| < \frac{\varepsilon}{2}$ for m large enough, and $\|U_m(t)\varphi - \varphi\| < \frac{\varepsilon}{2}$ for t small enough. So $\|U(t)\varphi - \varphi\|$ gets arbitrarily small as $t \to 0$, and $U(t)\varphi \to \varphi$.

Finally, A being a generator of $U(t)$ means that $A\varphi = \lim_{t\to 0} i\frac{U(t)\varphi-\varphi}{t}$ for any $\varphi \in D$. Now, by the fundamental theorem of calculus and since A_m generates $U_m(t)$,

$$i\frac{U(t)\varphi - \varphi}{t} = \lim_{m\to\infty} i\frac{U_m(t)\varphi - \varphi}{t} = \lim_{m\to\infty} \frac{1}{t}\int_0^t U_m(s)A_m\varphi \mathrm{d}s \ . \tag{30}$$

We now show that the right-hand side amounts to $\frac{1}{t}\int_0^t U(s)A\varphi \mathrm{d}s$, which converges to $A\varphi$ as $t \to \infty$, and would thus finish the proof.

$$\begin{aligned}
&\left\|\lim_{m\to\infty} \frac{1}{t}\int_0^t U_m(s)A_m\varphi \mathrm{d}s - \frac{1}{t}\int_0^t U(s)A\varphi \mathrm{d}s\right\| \\
\leq&\frac{1}{t}\int_0^t \lim_{m\to\infty} \|U_m(s)A_m\varphi - U(s)A\varphi\|\mathrm{d}s \\
\leq&\frac{1}{t}\int_0^t \lim_{m\to\infty} (\|A_m\varphi - A\varphi\| + \|U_m(s)A\varphi - U(s)A\varphi\|)\mathrm{d}s = 0 \ ,
\end{aligned} \tag{31}$$

as the term in the bracket vanishes as $m \to \infty$. Here, we were able to exchange integral and limit because of the dominated convergence theorem, which holds since

$$\begin{aligned}
&\|A_m\varphi - A\varphi\| + \|U_m(s)A\varphi - U(s)A\varphi\| \\
\leq&\|A_m\varphi - A\varphi\| + \big(\|U_m(s)\| + \|U(s)\|\big)\|A\varphi\| \\
=&\|A_m\varphi - A\varphi\| + 2\|A\varphi\|
\end{aligned} \tag{32}$$

is uniformly bounded in m. This finishes the proof. □

Problem 5: Resolvent of the Laplacian (10 points)

We have to show that for $H_0 = -\Delta$, $\varphi \in L^2(\mathbb{R}^3)$ and $\kappa > 0$, we have

$$\left(\left(H_0 + \kappa^2\right)^{-1}\varphi\right)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|}\varphi(y)\mathrm{d}y \ . \tag{33}$$

The action of $(H_0 + \kappa^2)^{-1}$ is best described in Fourier space, where it amounts to a multiplication of $\hat{\varphi}(k)$ by $f(k) := \frac{1}{|k|^2+\kappa^2}$. Thus, by the convolution theorem,

$$\left(\left(H_0 + \kappa^2\right)^{-1}\varphi\right)(x) = (f\hat{\varphi})^\vee(x) = (2\pi)^{-\frac{3}{2}}(\check{f} * \varphi)(x) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^n} \check{f}(x-y)\varphi(y)\mathrm{d}y \ . \tag{34}$$

Thus, it remains to show that $(2\pi)^{-\frac{3}{2}}\check{f}(x) = \frac{e^{-\kappa|x|}}{4\pi|x|}$ in order to finish the proof. This is done by evaluating the Fourier transform in spherical coordinates, using the substitution $y = \cos\theta$:

$$\begin{aligned}
&(2\pi)^{-\frac{3}{2}}\check{f}(x) = (2\pi)^{-3}\int_{\mathbb{R}^3} f(k)e^{ikx}\mathrm{d}k = (2\pi)^{-3}\int_0^\infty\int_0^\pi \frac{e^{ir|x|\cos\theta}}{r^2+\kappa^2}2\pi r^2\sin\theta\mathrm{d}\theta\mathrm{d}r \\
=&(2\pi)^{-2}\int_0^\infty\int_{-1}^1 \frac{e^{ir|x|y}}{r^2+\kappa^2}r^2\mathrm{d}y\mathrm{d}r = (2\pi)^{-2}\int_0^\infty \frac{e^{ir|x|} - e^{-ir|x|}}{i|x|(r^2+\kappa^2)}r\mathrm{d}r \\
=&(2\pi)^{-2}\Im\int_0^\infty \frac{2re^{ir|x|}}{|x|(r^2+\kappa^2)}\mathrm{d}r =: (2\pi)^{-2}\Im\int_0^\infty g(r)\mathrm{d}r \ .
\end{aligned} \tag{35}$$

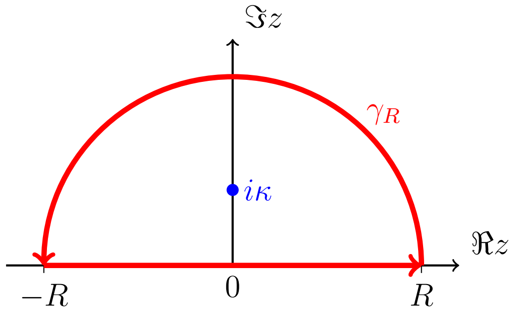

Figure 1: The integration contour around the pole at $i\kappa$.

We can now interpret $g(r)$ as a complex function $g : \mathbb{C} \to \mathbb{C}$, which has two poles at $r = \pm i\kappa$. This allows us to exploit the residue theorem: Let $R > \kappa$ and define the integration contour $\gamma_R \subset \mathbb{C}$, which runs from $-R$ to R along the real axis and then closes in a semi-circle around 0 with radius R, see Figure 1. Then, by the residue theorem,

$$\begin{aligned}\oint_{\gamma_R} g(z)\mathrm{d}z &= 2\pi i \operatorname{Res}(g, i\kappa) = 2\pi i \lim_{z\to i\kappa}(z - i\kappa)g(z) \\ &= 2\pi i \lim_{z\to i\kappa} \frac{2ze^{iz|x|}}{|x|(z + i\kappa)} = 2\pi i \frac{e^{-\kappa|x|}}{|x|} \ .\end{aligned} \tag{36}$$

On the other hand, as $R \to \infty$, we get

$$\lim_{R\to\infty} \oint_{\gamma_R} g(z)\mathrm{d}z = \int_{-\infty}^{\infty} g(r)\mathrm{d}r + \lim_{R\to\infty} \int_0^{\pi} g(Re^{i\theta})Rie^{i\theta}\mathrm{d}\theta \tag{37}$$

Now,

$$|g(Re^{i\theta})| \le \frac{2R}{|x|(R^2 - \kappa^2)} \to 0 \qquad \text{as } R \to \infty \ , \tag{38}$$

so the second integral vanishes due to Jordan's lemma and we are left with

$$\int_{-\infty}^{\infty} g(r)\mathrm{d}r = 2\pi i \frac{e^{-\kappa|x|}}{|x|} \ . \tag{39}$$

To apply this result to (35), we still need to extend the integral $\int_0^\infty g(r)\mathrm{d}r$ from (35) to the real line, which requires adding

$$\int_{-\infty}^{0} g(r)\mathrm{d}r = \int_0^{\infty} g(-r)\mathrm{d}r = \int_0^{\infty} g(r)\mathrm{d}r \tag{40}$$

We conclude:

$$2(2\pi)^{-\frac{3}{2}}\check{f}(x) = (2\pi)^{-2}\Im \int_{-\infty}^{\infty} g(r)\mathrm{d}r = 2\frac{e^{-\kappa|x|}}{4\pi|x|} \ , \tag{41}$$

which renders the desired result $(2\pi)^{-\frac{3}{2}}\check{f}(x) = \frac{e^{-\kappa|x|}}{4\pi|x|}$. □