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Metodi Matematici della Meccanica Quantistica

Assignment 1

To be handed in on **Wednesday, October 4, 2023, before 10:30** via email (scanned or LaTeX) to sascha.lill@unimi.it or on paper at the beginning of the lecture.

Problem 1: Banach Spaces and Bounded Operators (2+2+3+3 points)

a. Show that the following is a norm on $L^p(\mathbb{R}^d)$:

$$\|[f]\|_p := \left(\int |f(x)|^p \mathrm{d}x \right)^{1/p} = \left(\int \mathrm{d}x \, |f(x)|^p \right)^{1/p} .$$

b. Let X be a Banach space and $\tilde{X} \subset X$ a closed subspace. Show that $\tilde{X}$ is a Banach space.

c. Let X be a normed vector space and Y be a Banach space. Show that the space of bounded linear operators from X to Y, i. e., $\mathcal{L}(X, Y)$ with the operator norm, is a Banach space.

d. Let X, Y be Banach spaces and $A : D \to Y$ a bounded linear operator defined on a dense subspace $D \subset X$. Construct an extension $\overline{A}$ of A to all of X such that the extension is a linear operator with the same operator norm as A, i. e., $\|\overline{A}\| = \|A\|$.

Problem 2: Derivative Operator (5+5 points)

Consider (as in the example presented in the lecture) the Banach space $X = C([0, 1])$ of continuous functions with the norm $\|f\| := \sup_{x \in [0,1]} |f(x)|$.

Define operators $A_k : D_k \subset X \to X$ (for the two options $k \in \{3, 4\}$) by the mapping

$$(A_k f)(x) := f'(x)$$

with domains given as subsets of the continuously differentiable functions as follows

$$D_3 := \{f \in C^1([0, 1]) \mid f(0) = f(1)\} \qquad \text{(periodic boundary conditions)} ,$$
$$D_4 := \{f \in C^1([0, 1]) \mid f(0) = 0 = f(1)\} \qquad \text{(Dirichlet boundary conditions)} .$$

Show that

a. $\sigma(A_3) = \sigma_{\mathrm{p}}(A_3) = 2\pi i \mathbb{Z}$,

b. $\sigma(A_4) = \mathbb{C}$ and $\sigma_{\mathrm{p}}(A_4) = \emptyset$.

Problem 3: Operator-valued analytic functions (10 points)

Recall Liouville's theorem of complex analysis: Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function (defined and analytic on *all* of $\mathbb{C}$) and bounded, i. e., there exists $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is constant.

Show that: Let X be a Banach space. Let $L : \mathbb{C} \to \mathcal{L}(X)$ be an operator-valued analytic function (as defined in the lecture) on all $\mathbb{C}$. If L is bounded in the sense that there exists $M \in \mathbb{R}$ such that $\|L(z)\| \leq M$ for all $z \in \mathbb{C}$, then L is constant.