Soluzione Esercizio VII

Problema 1: (1) due radici coincidenti:

$$x^2+2x+1=(x+1)^2$$
$\Rightarrow$ radici: $x_{1,2}=-1$ (due coincidenti)

$$\cancel{\frac{A}{x-x_1}} \quad \cancel{\frac{B}{x-x_2}}$$

$$\frac{1x+0}{x^2+2x+1}=\frac{A}{x+1}+\frac{B}{(x+1)^2}=\frac{A(x+1)+B}{(x+1)^2}=\frac{Ax+(A+B)}{(x+1)^2}$$

$$\Rightarrow \quad A=1, \quad A+B=0 \quad \Rightarrow \quad A=1, \quad B=-1.$$

$$\Rightarrow \int \frac{x}{x^2+2x+1}\,dx = A\int\frac{1}{x+1}\,dx + B\int\frac{1}{(x+1)^2}\,dx$$
$$=\int\frac{1}{x+1}\,dx-\int\frac{1}{(x+1)^2}\,dx$$
$$=\ln|x+1|-\left(-\frac{1}{(x+1)}\right)+C$$

Controllo: $\frac{d}{dx}\left(-\frac{1}{x+1}\right)=-\frac{d}{dx}(x+1)^{-1}$
$$=-(-1)(x+1)^{-2}\cdot 1=\frac{1}{(x+1)^2}.$$

(2) <u>Denominatore senza radici reali:</u>

$$\int\frac{1-2x}{x^2+2x+5}\,dx$$

Radici: $x^2+2x+5=0$

$$x_{1,2}=\frac{-2\pm\sqrt{4-4\cdot5}}{2} \qquad 4-4\cdot5<0$$

$\Rightarrow$ radici non sono reali.

$$\frac{1-2x}{x^2+2x+5}=\frac{A\,(\text{derivata del denominatore})+B}{x^2+2x+5}$$
$$=\frac{A(2x+2)+B}{x^2+2x+5}$$

$$\frac{d}{dx}(x^2+2x+5)=2x+2$$

$$1-2x=2Ax+(2A+B) \quad\Rightarrow\quad A=-1$$
$$\left.\begin{aligned}2A+B&=1\\ -2+B&=1 \quad\Rightarrow\quad B=3.\end{aligned}\right\}$$

$$\Rightarrow\int\frac{1-2x}{x^2+2x+5}\,dx=\underbrace{(-1)}_{A}\int\frac{2x+2}{x^2+2x+5}\,dx+\underbrace{3}_{B}\int\frac{1}{x^2+2x+5}\,dx$$

"tipo funzione composta"

$$= -\ln|x^2+2x+5| + C' + 3\underbrace{\int \frac{1}{x^2+2x+5}\,dx}$$

$$\int \frac{1}{\underbrace{x^2+2x+5}_{(x+1)^2}}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\,dx$$

Sostituzione:
$y = \frac{x+1}{2}$
$x = 2y-1$
$dx = 2\,dy$

$$\overset{\text{Sostituzione}}{=} \frac{1}{4}\int \frac{1}{y^2+1}\,2\,dy = \frac{1}{2}\operatorname{arctg}(y) + C'$$

$y = \frac{x+1}{2}$

$$= \frac{1}{2}\operatorname{arctg}\left(\frac{x+1}{2}\right) + C'.$$

$$\Rightarrow \int \frac{1-2x}{x^2+2x+5}\,dx = -\ln|x^2+2x+5| + \frac{3}{2}\operatorname{arctg}\left(\frac{x+1}{2}\right) + C.$$

Domanda: $\int \frac{1}{x^2+2x+5}\,dx \overset{?}{=} \ln|x+1|^2$

$$D\ln\left(|x+1|^2\right) = \frac{1}{|x+1|^2}\cdot \underline{2(x+1)}$$

$$D\ln(x^2+2x+5) = \frac{1}{x^2+2x+5}\cdot 2x$$

Attenzione!

(3) $\int \frac{t^3+2t^2+t}{t^2-2t+1}\,dt$ (a lezione maggio 13, 5).

Divisione tra polinomi: (perché grado del numeratore > grado denominatore)

$$t^3+2t^2+t = (t^2-2t+1)(t+4) + 8t-4$$

$$\begin{array}{l} t^3+2t^2+t \\ -(t^3-2t^2+t) \\ \hline 4t^2 \\ -(4t^2-8t+4) \\ \hline +8t-4 \end{array}$$

$4(2t-1)$

$$\Rightarrow \int \frac{t^3+2t^2+t}{t^2-2t+1}\,dt = \underbrace{\int (t+4)\,dt}_{=\frac{1}{2}t^2+4t} + \int \frac{8t-4}{t^2-2t+1}\,dt$$

una radice doppia!

Come problema 1.(1). ③

$$\frac{8t-4}{t^2-2t+1} = \frac{A}{t-1} + \frac{B}{(t-1)^2}$$

$$8t-4 = A(t-1) + B = At + (B-A)$$

$$\Rightarrow A = 8 \qquad B - A = -4$$

$$B - 8 = -4 \Rightarrow \boxed{B = 4}$$

$$\Rightarrow \int \frac{8t-4}{t^2-2t+1}\,dt = 8\int \frac{1}{t-1}\,dt + 4\int \frac{1}{(t-1)^2}\,dt$$

$$= 8\ln|t-1| + 4(-1)\frac{1}{t-1}$$

$$= 8\ln|t-1| - 4\frac{1}{t-1}.$$

Controllo: $D\left(8\ln|t-1| - 4\frac{1}{t-1}\right)$

$$= 8\frac{1}{t-1} - 4\,D(t-1)^{-1}$$

$$= 8\frac{1}{t-1} - 4(-1)(t-1)^{-2}$$

$$= \frac{8}{t-1} + \frac{4}{(t-1)^2} = \frac{8(t-1)+4}{(t-1)^2}$$

$$= \frac{8t-4}{(t-1)^2}.$$

$$\frac{d}{dx}x^{-1} = -1\,x^{-2}$$

$$\frac{d}{dx}\frac{1}{x} = +\frac{1}{x^2}$$

$$-\int \frac{d}{dx}\frac{1}{x}\,dx = \int \frac{1}{x^2}\,dx$$

$$\parallel$$

$$-\frac{1}{x}$$

Problema 2: (1) Int. per parti, integrale definito:

$$\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Soluzione: Visto a lezione:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

$$\int_a^b f(x)g'(x)\,dx = \left[\int f(x)g'(x)\,dx\right]_a^b = \left[f(x)g(x) - \int f'(x)g(x)\,dx\right]_a^b$$

$$= \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx.$$

(2) Int. per sostituzione, integrale indefinito:

$$\int f(g(t))\, g'(t)\, dt = \int f(x)\, dx \Big|_{x=g(t)}.$$

Soluzione: Sia F una primitiva di f: $\frac{dF}{dx}(x) = f(x)$.

$$\frac{d}{dt}\left(\int f(x)\, dx \Big|_{x=g(t)}\right) = \frac{d}{dt}\left(F(g(t)) + C\right)$$

$$= F'(g(t))\, g'(t) = f(g(t))\, g'(t).$$

$\blacksquare$

(3) Int. per sost., integrale definito:

$$\int_{t_0}^{t_1} f(g(t))\, g'(t)\, dt = \int_{g(t_0)}^{g(t_1)} f(x)\, dx.$$

Dimostrazione:

$$\int_{t_0}^{t_1} f(g(t))\, g'(t)\, dt = \left[\int f(g(t))\, g'(t)\, dt\right]_{t_0}^{t_1}$$

$$= \left[F(g(t))\right]_{t_0}^{t_1} = \left[F(x)\right]_{g(t_0)}^{g(t_1)} = \left[\int f(x)\, dx\right]_{g(t_0)}^{g(t_1)}$$

$$= \int_{g(t_0)}^{g(t_1)} f(x)\, dx.$$

$\blacksquare$

Problema 3:

(Non facciamo tutti gli integrali, solo due o tre di ogni tipo.)

- $$\int \underset{f}{x}\, \underset{g'}{e^x}\, dx \overset{\text{int. per parti}}{=} \underset{fg}{x e^x} - \underset{\int f'g}{\int e^x\, dx} = x e^x - e^x + C.$$

$$f(x) = x, \quad f'(x) = 1, \quad g'(x) = e^x, \quad g(x) = e^x$$

- $$\int \frac{\ln(x)}{x^5}\, dx = \int \underset{f}{\ln(x)}\, \underset{g'}{x^{-5}}\, dx$$

$$f(x) = \ln(x), \quad f'(x) = 1/x, \quad g'(x) = x^{-5}, \quad g(x) = -\frac{1}{4} x^{-4}$$

$$= \underset{fg}{\ln(x)\left(-\frac{1}{4}x^{-4}\right)} - \underset{\int f'g}{\int \frac{1}{x}\left(-\frac{1}{4}x^{-4}\right) dx}$$

$$= -\frac{1}{4}\ln(x)\frac{1}{x^4} + \frac{1}{4}\underbrace{\int \frac{1}{x^5}\, dx}_{= -\frac{1}{4}x^{-4}} = -\frac{1}{4}\ln(x)\frac{1}{x^4} - \frac{1}{16}\frac{1}{x^4} + C.$$

- $\int \underset{f'}{2x}\ \underset{g}{\operatorname{arctg}(x)}\ dx$ — derivata qui!

integrazione per parti

$f'(x) = 2x$
$f(x) = x^2$
$g(x) = \operatorname{arctg}(x)$
$g'(x) = \frac{1}{1+x^2}$

$$= x^2 \operatorname{arctg}(x) - \int x^2 \frac{1}{1+x^2}\,dx$$
$$f g \quad - \int f g'\,dx$$
$$= x^2 \operatorname{arctg}(x) - \int \frac{x^2+1-1}{x^2+1}\,dx$$
$$= x^2 \operatorname{arctg}(x) - \int \left(1 - \frac{1}{x^2+1}\right)dx$$
$$= x^2 \operatorname{arctg}(x) - x + \underbrace{\int \frac{1}{x^2+1}\,dx}_{= \operatorname{arctg}(x) + C}$$
$$= x^2 \operatorname{arctg}(x) - x + \operatorname{arctg}(x) + C.$$

Controllo: $\frac{d}{dx}\left(x^2 \operatorname{arctg}(x) - x + \operatorname{arctg}(x)\right)$
$$= 2x \operatorname{arctg}(x) + x^2 \frac{1}{1+x^2} - 1 + \frac{1}{1+x^2}$$
$$= 2x \operatorname{arctg}(x) + \underbrace{\frac{x^2+1}{1+x^2} - \frac{1+x^2}{1+x^2}}_{=0} = 2x \operatorname{arctg}(x). \quad \checkmark$$

- $\lim_{n \to +\infty} \int_1^n \frac{kx+k}{x^3}\,dx \qquad (k \in \mathbb{R})$

$$\int_1^n \frac{kx+k}{x^3}\,dx = k \int_1^n \frac{x+1}{x^3}\,dx = k \int_1^n \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx$$
$$= k\left(\left[(-1)x^{-1}\right]_1^n + \left[\frac{1}{-2}x^{-2}\right]_1^n\right)$$
$$= k\left(-\frac{1}{n} + 1 - \frac{1}{2n^2} - \frac{1}{-2}\right) = k\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{2n^2}\right).$$
$$\lim_{n\to\infty} k\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{2n^2}\right) = k\frac{3}{2}.$$

- $\int \operatorname{arcsen}(5x)\,dx$

Sostituzione: $5x = y$
$x = \frac{1}{5}y$
$dx = \frac{1}{5}dy$

$$= \frac{1}{5}\int \underset{f}{\operatorname{arcsen}(y)}\ \underset{g'}{dy}$$

Int. per parti: $f(y) = \arcsin(y)$ $\quad g'(y) = 1$

$f'(y) = \frac{1}{\sqrt{1-y^2}}$ $\quad g(y) = y$

$$= \frac{1}{5}\left(\arcsin(y)\, y - \int \frac{1}{\sqrt{1-y^2}}\, y \, dy \right) \qquad (*)$$

$$f g - \int f' g$$

per l'integrale $\int \frac{1}{\sqrt{1-y^2}}\, y\, dy$: di nuovo sostituzione:

$t = y^2 \qquad dt = \frac{d}{dy}(y^2)\, dy$
$= 2y\, dy$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-t}}\, dt = \frac{1}{2}\int (1-t)^{-1/2}\, dt$$

funzione composta

$$\int x^{-1/2}\, dx = \frac{1}{\left(\frac{1}{2}\right)} x^{1/2} + C = 2x^{1/2} + C$$

fare il controllo!

$$= \frac{1}{2}\, \frac{1}{1/2} (1-t)^{1/2}\, \underline{(-1)} + C$$

$$= -\sqrt{1-t} + C.$$

Sostituzione inversa: $t = y^2$

$$= -\sqrt{1-y^2} + C \quad \Rightarrow \quad \int \frac{y}{\sqrt{1-y^2}}\, dy = -\sqrt{1-y^2} + C$$

inserire in $(*)$:

$$\frac{1}{5}\left(\arcsin(y)\, y + \sqrt{1-y^2} + C \right)$$

Sostituzione inversa: $y = 5x$

$$\Rightarrow \int \arcsin(5x)\, dx = \frac{1}{5}\left(\arcsin(5x)\, 5x + \sqrt{1-25x^2} + C \right).$$

- $\int \underline{x^2}\, e^{x^3}\, \underline{dx}$: Sostituzione: $x^3 = y$

$dy = \frac{d}{dx}(x^3)\, dx$
$= 3\underline{x^2\, dx}$

$x^2\, dx = \frac{1}{3}\, dy$

$$= \int e^y \, \frac{1}{3}\, dy$$

$$= \frac{1}{3} e^y = \frac{1}{3} e^{(x^3)}.$$

- $\displaystyle\int_2^{\pi/6} \frac{e^x}{1-e^x}\,dx$

$\underbrace{y = 1-e^x}$ oppure $y = e^x$

$$dy = \frac{d}{dx}(1-e^x)\,dx = -e^x\,dx$$

$$\Leftrightarrow \quad e^x\,dx = -dy$$

$$= -\int_{1-e^2}^{1-e^{\pi/6}} \frac{1}{y}\,dy$$

Estremi dell'intervallo di integrazione:

$x_0 = 2 \Rightarrow y_0 = (1-e^{x_0}) = (1-e^2)$

$x_1 = \pi/6 \Rightarrow y_1 = (1-e^{\pi/6})$

$$= \Big[-\ln|y|\Big]_{1-e^2}^{1-e^{\pi/6}}$$

$$= -\ln|1-e^{\pi/6}| + \ln|1-e^2|$$

$$= \ln\left|\frac{1-e^2}{1-e^{\pi/6}}\right|$$

LEZIONE

Integrali impropri

Integrali su intervalli aperti alla destra o sinistra, o su intervalli unlimitati.

Esempi:

$\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ intervallo illimitato $[1,+\infty)$

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ funzione definita solo su $(0,1]$

$$\lim_{x\to 0+} \frac{1}{\sqrt{x}} = +\infty.$$

Definizione: Integrali impropri

$$\int_a^{+\infty} f(x)\,dx := \lim_{M\to+\infty} \int_a^M f(x)\,dx \qquad a \in \mathbb{R}$$

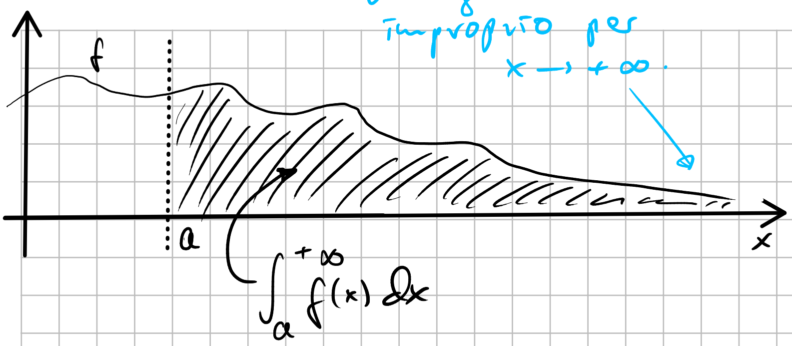
$$\int_b^c f(x)\,dx := \lim_{h\to b+} \int_h^c f(x)\,dx. \qquad b, c \in \mathbb{R},\ b < c$$

se $\displaystyle\lim_{x\to b+} f(x) = \pm\infty$

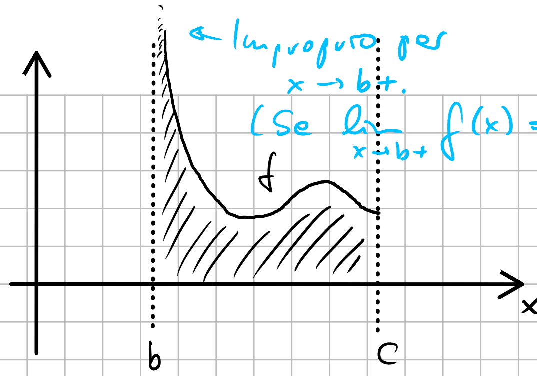
Si dice: l'integrale improprio è convergente se il limite esiste, e divergente se il limite non esiste.

con valore finito!

L'integrale è improprio per $x \to +\infty$.



← Improprio per $x \to b^+$. (Se $\lim_{x \to b^+} f(x) = \pm\infty$).



$\boxed{\text{Esempi:}}$

- $$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{M \to \infty} \int_1^M \frac{1}{x^2}\,dx = \lim_{M \to \infty} \left[-1\,\frac{1}{x}\right]_1^M$$
$$= \lim_{M \to \infty} \left(-\frac{1}{M} - \left(-\frac{1}{1}\right)\right) = \lim_{M \to \infty} \left(1 - \frac{1}{M}\right) = 1.$$

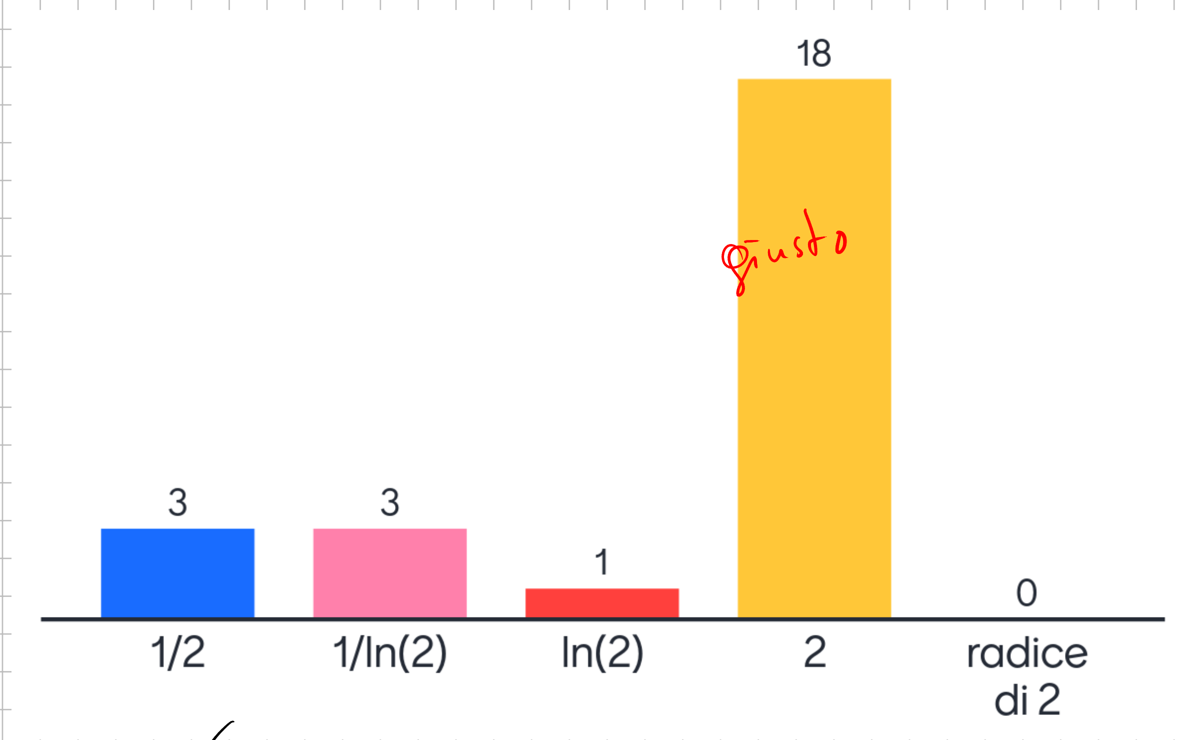
L'integrale improprio è convergente.

- $$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{M \to +\infty} \Big[\ln|x|\Big]_1^M = \lim_{M \to +\infty} (\ln(M) - 0)$$
$$= \lim_{M \to \infty} \ln(M) = +\infty.$$

L'integrale improprio è divergente.

$\boxed{\text{Menti:}}$ $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = 2$$ (in particolare $< +\infty$, allora convergente.)

$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty \Rightarrow$ L'integrale è improprio per $x \to 0^+$.



Soluzione

In generale: (il mento era il caso $p = \frac{1}{2}$)

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-p}\, dx = \lim_{\varepsilon \to 0^+} \left[\frac{x^{1-p}}{1-p} \right]_\varepsilon^1$$

per $p \neq 1$

(per $p = 1$ attenzione: logaritmo!)

$$= \lim_{\varepsilon \to 0^+} \left(\frac{1}{1-p} - \frac{\varepsilon^{1-p}}{1-p} \right) = \frac{1}{1-p}$$

per $p < 1$.

Invece per $p > 1$: non è convergente.

$\blacksquare$

Da ricordare:

$$\int_0^1 \frac{1}{x^p}\, dx \begin{cases} < +\infty & p < 1 \\ = +\infty & p \geq 1 \end{cases}$$

Per $x \to 0$, l'integrale è convergente se p è piccolo.

$$\int_1^{+\infty} \frac{1}{x^p}\, dx \begin{cases} = +\infty & p \leq 1 \\ < +\infty & p > 1. \end{cases}$$

Per $x \to +\infty$, l'integrale è convergente se p è grande.