Describing Quantum Correlations in the Fermi Liquid by Bosonization

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Electrons in a Piece of Metal

Hamilton operator of N spinless fermions on the (fixed size) 3D torus:

$$H_N := \sum_{i=1}^{N} \left(-\hbar^2 \Delta_i\right) + \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \,, \quad V : \mathbb{R}^3 \to \mathbb{R} \,.$$

Acts on the L^2–space of antisymmetric wave functions of $3N$ variables.

Quantities of interest: spectrum and ground state energy:

$$E_N := \inf \sigma(H_N) = \inf_{\|\psi\|=1} \langle \psi, H_N \psi \rangle \,.$$

Fixing the physical regime of interest: gas with high density & weak interactions,
$\hbar := N^{-1/3}$, $\lambda := N^{-1}$, and $N \to \infty$ (mean–field/semiclassical scaling limit).

Bosonization of Excitations — Definition and Project Goals

Preparation: Isolating Excitations

$$H_N = \hbar^2 \sum_{k\in\mathbb{Z}} |k|^2 a_k^* a_k + \frac{1}{2N} \sum_{k\in\mathbb{Z}^3} \hat{V}(k) \sum_{p,q\in\mathbb{Z}^3} a_{p+k}^* a_{q-k}^* a_q a_p \;.$$

Particle–hole transformation corresp. to Fermi ball $\mathcal{B}_F = \{k \in \mathbb{Z}^3 : |k| \leq (\frac{3}{4\pi})^{1/3} N^{1/3}\}$:

$$R\, a_k^*\, R^* := \begin{cases} a_k^* & k \in \mathcal{B}_F^c \\ a_k & k \in \mathcal{B}_F \;. \end{cases}$$

We get

$$R^* H_N R = E_N^{\mathrm{HF}} + \underbrace{\hbar^2 \sum_{p\in\mathcal{B}_F^c} p^2 a_p^* a_p - \hbar^2 \sum_{h\in\mathcal{B}_F} h^2 a_h^* a_h}_{=:\; H^{\mathrm{kin}}} + \underbrace{Q}_{\text{quartic in operators } a^*,\, a}$$

Goal: a quadratic approximation to the excitation Hamiltonian $H^{\mathrm{kin}} + Q$.

(Quadratic Hamiltonians can be diagonalized by Bogoliubov transformations.)

Bosonization of the Interaction

Observe: if we introduce pair operators

$$b_k^* := \sum_{\substack{p \in \mathcal{B}_F^c \\ h \in \mathcal{B}_F}} \delta_{p-h,k} \; a_p^* a_h^*$$

p "particle" outside the Fermi ball
h "hole" inside the Fermi ball

then

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \Big(2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \Big) + \mathcal{O}(N^{-1}) \, .$$

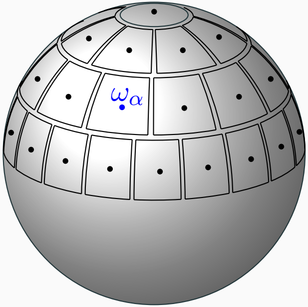
The b_k^* and b_k have approximately bosonic commutators:

$$[b_k^*, b_l^*] = 0 \, , \qquad [b_l, b_k^*] = \delta_{k,l} n_k^2 + \mathcal{E}(k,l) \, .$$

How to express H^{kin} through pair operators?

Bosonization of the Kinetic Energy

Fermi ball $\mathcal{B}_F$



[Benfatto–Gallavotti '90]
[Houghton–Marston '93]
[Haldane '94]
[Castro Neto–Fradkin '94]
[Fröhlich–Götschmann–Marchetti '95]

Localize to $M = M(N)$ patches near the Fermi surface:

$$b^*_{\alpha,k} := \frac{1}{n_{\alpha,k}} \sum_{\substack{p \in \mathcal{B}_F^c \cap B_\alpha \\ h \in \mathcal{B}_F \cap B_\alpha}} \delta_{p-h,k}\; a^*_p a^*_h$$

with $n_{\alpha,k}$ chosen to normalize $\|b^*_{\alpha,k}\Omega\| = 1$.

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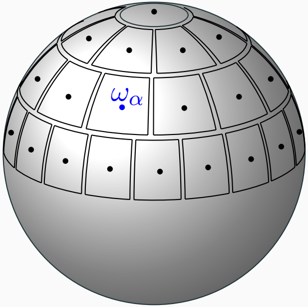

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Linearize around patch center ω_α:

$$[H^{\text{kin}}, b^*_{\alpha,k}] \simeq 2\hbar |k \cdot \hat{\omega}_\alpha| b^*_{\alpha,k} \;.$$

Same commutator obtained with the replacement

$$H^{\text{kin}} \rightsquigarrow \sum_{k \in \mathbb{Z}^3} \sum_{\alpha=1}^{M} 2\hbar u_\alpha(k)^2 b^*_{\alpha,k} b_{\alpha,k} \;, \qquad u_\alpha(k)^2 := |k \cdot \hat{\omega}_\alpha| \;.$$

Quadratic Effective Hamiltonian

Back to the interaction

$$Q \simeq \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left(2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \right) \; .$$

Decompose

$$b_k^* = \sum_{\alpha=1}^{M} n_{\alpha,k} b_{\alpha,k}^* + \text{lower order} \; .$$

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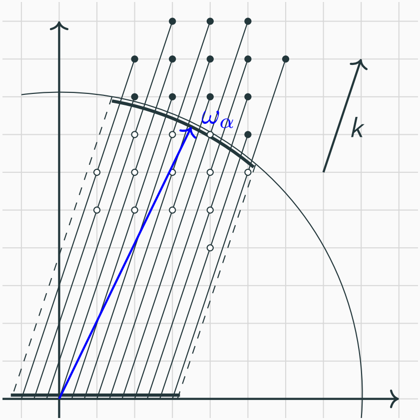
Normalization:

$$n_{\alpha,k}^2 = \#\text{p-h pairs in patch } B_\alpha \text{ with momentum } k$$

$$\simeq \frac{4\pi N^{2/3}}{M} |k \cdot \hat{\omega}_\alpha| \ .$$



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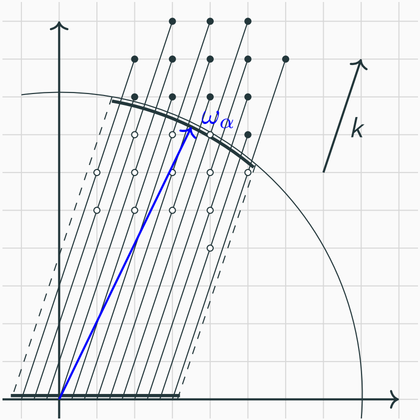


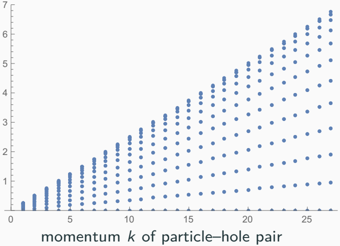
Effective Quadratic Almost–Bosonic Hamiltonian

$$H^{\text{eff}} = \hbar \sum_{k \in \mathbb{Z}^3} \left[\sum_{\alpha} u_\alpha(k)^2 b_{\alpha,k}^* b_{\alpha,k} + \frac{\hat{V}(k)}{M} \sum_{\alpha,\beta} \left(u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,k} + u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,-k}^* + \text{h.c.} \right) \right]$$

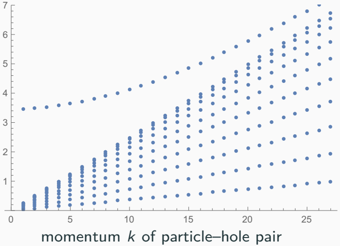
Diagonalization of the Effective Hamiltonian

In the bosonic approximation $\mathcal{E}(k,l) = 0$, H^{eff} can be diagonalized by a bosonic Bogoliubov transformation [B, Rev. Math. Phys. (2020)]:

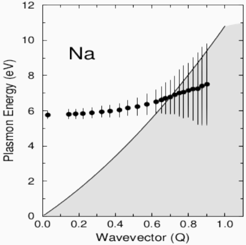
no interaction



Coulomb $\hat{V}(k) = 1/|k|^2$



Sodium (experimental)



Plasmon (collective oscillation) emerges — rest of spectrum only weakly renormalized.

a non–perturbative approach to Fermi liquid theory

State of the Rigorous Project

Ground State Energy

Theorem: [B–Nam–Porta–Schlein–Seiringer, Commun. Math. Phys. (2020) and Invent. Math. (2021), B–Porta–Schlein–Seiringer arXiv:2106.13185]

Let $\hat{V}(k) \geq 0$ and $\sum_{k \in \mathbb{Z}^3} (1 + |k|) \hat{V}(k) < +\infty$. Then as $N \to \infty$ we have

$$E_N = E_N^{\mathrm{HF}} + E_N^{\mathrm{RPA}} + \mathcal{O}(\hbar^{1+\varepsilon}) \qquad (\hbar = N^{-1/3}) ,$$

where the random phase approximation energy is

$$E_N^{\mathrm{RPA}} := \hbar \sum_{k \in \mathbb{Z}^3} |k| \left[\int_0^\infty \log\left(1 + \hat{V}(k)\Big(1 - \lambda \arctan \lambda^{-1}\Big)\right) \mathrm{d}\lambda - \frac{1}{4} \hat{V}(k) \right] .$$

- $E_N^{\mathrm{RPA}} \simeq \inf \sigma(H^{\mathrm{eff}})$, as formally computed by the Bogoliubov diagonalization.
- In physics (1950s): Macke, Bohm–Pines, Gell-Mann–Brueckner, Sawada et al
- Also: Christiansen–Hainzl–Nam arXiv:2106.11161

Our bosonization method has potential far beyond the ground state energy: spectrum, ground state properties, dynamics...

Example: Effective Dynamics

2014 In the sense of reduced density matrices, Hartree–Fock theory is sufficient to approximate the many–body time evolution:

$$\|\gamma_t^{(1)} - \gamma_t^{\mathrm{HF}}\|_{\mathrm{tr}} \leq \frac{\exp(c_1 \exp(c_2|t|))}{N^{5/6}} \, .$$

[B–Porta–Schlein, Commun. Math. Phys. (2014)]

2021 An effective bosonized dynamics provides a much stronger approximation, i. e., in Fock space norm of the many–body wave function.

[B–Nam–Porta–Schlein–Seiringer arXiv:2103.08224]