# Correlation Energy of a Weakly–Interacting Fermi Gas

Niels Benedikter

joint work with Phan Thành Nam, Marcello Porta, Benjamin Schlein, and Robert Seiringer

Università degli Studi di Milano

# Quantum System of $N$ Fermions

Hamilton operator of $N$ identical spinless particles on the (fixed size) 3D torus:

$$H_N := \sum_{i=1}^{N} (-\Delta_i) + \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \qquad \text{with } V : \mathbb{R}^3 \to \mathbb{R} \, .$$

Acts on the $L^2$–space of antisymmetric wave functions of $3N$ variables

$$\psi(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) = \operatorname{sgn}(\sigma) \psi(x_1, x_2, \ldots, x_N) \qquad \forall \sigma \in S_N \, .$$

For reasonable potentials, the Hamiltonian is self–adjoint.

$\operatorname{spec}(H_N)$ is interpreted as excitation energies measurable in experiments.

# Ground State Energy

The ground state energy is defined as

$$E_N := \inf \operatorname{spec}(H_N) = \inf_{\substack{\psi \in L^2_a(\mathbb{T}^{3N}) \\ \|\psi\|=1}} \langle \psi, H_N \psi \rangle \,.$$

How to compute $E_N$? Define the reduced density matrices

$$\gamma^{(2)} := \frac{N!}{(N-2)!} \operatorname{tr}_{3,4,\ldots N} |\psi\rangle\langle\psi| \,, \qquad \gamma^{(1)} := \frac{1}{N-1} \operatorname{tr}_2 \gamma^{(2)} \,.$$

Then

$$\langle \psi, H_N \psi \rangle = \operatorname{tr}\left(-\Delta \gamma^{(1)}\right) + \frac{1}{2} \iint V(x_1 - x_2) \gamma^{(2)}(x_1, x_2; x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \,.$$

# Ground State Energy

The ground state energy is defined as

$$E_N := \inf \operatorname{spec}(H_N) = \inf_{\substack{\psi \in L^2_a(\mathbb{T}^{3N}) \\ \|\psi\|=1}} \langle \psi, H_N \psi \rangle \,.$$

How to compute $E_N$? Define the reduced density matrices

$$\gamma^{(2)} := \frac{N!}{(N-2)!} \operatorname{tr}_{3,4,\dots N} |\psi\rangle\langle\psi| \,, \qquad \gamma^{(1)} := \frac{1}{N-1} \operatorname{tr}_2 \gamma^{(2)} \,.$$

Then

$$\langle \psi, H_N \psi \rangle = \operatorname{tr}\left(-\Delta \gamma^{(1)}\right) + \frac{1}{2} \iint V(x_1 - x_2) \gamma^{(2)}(x_1, x_2; x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \,.$$

So we simply minimize over $\gamma^{(2)}$?

The set of all 2-particle rdm is hard to characterize: N-representability problem.

# Mean–Field Regime

This problem cannot be solved in full generality: $H_N$ describes almost the entire variety of our daily lives, superconductors, neutron stars, our bodies...

Be more specific, look at well–defined physical situations!

## Mean–Field Regime

This problem cannot be solved in full generality: $H_N$ describes almost the entire variety of our daily lives, superconductors, neutron stars, our bodies...

Be more specific, look at well–defined physical situations!

Simplest possibility: high density and with weak interaction.
$\rightsquigarrow$ We expect mean–field behavior: one particle moving through a continuous cloud generated by all the other particles.

# Mean–Field Regime

This problem cannot be solved in full generality: $H_N$ describes almost the entire variety of our daily lives, superconductors, neutron stars, our bodies. . .

Be more specific, look at well–defined physical situations!

Simplest possibility: high density and with weak interaction.
$\rightsquigarrow$ We expect mean–field behavior: one particle moving through a continuous cloud generated by all the other particles.

### Mathematical Model: Mean–Field Scaling Regime

- high density: fixed volume (torus) and $N$ particles, $N \to \infty$.
- weak interation: $\lambda = N^{-1/3}$ because

$$\Big\langle \sum_{i=1}^{N} (-\Delta_i) \Big\rangle \sim N^{5/3} \quad \text{(antisymmetry!)}\,, \qquad \Big\langle \lambda \sum_{1\leq i<j\leq N} V(x_i - x_j) \Big\rangle \sim \lambda N^2\,.$$

# Leading Order Approximation: Hartree–Fock Theory

# Hartree–Fock Theory = Restriction to Slater Determinants

Multiply the entire Hamiltonian by $\times\hbar^2$, with $\hbar := N^{-1/3}$:

$$H_N = \sum_{i=1}^{N} \left(-\hbar^2 \Delta_i\right) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V(x_i - x_j) \ .$$

Convergence to the Hartree–Fock energy [Bach '92, Graf–Solovej '94]:

$$\left|E_N - E_N^{\mathrm{HF}}\right| = o(N)\,, \qquad \text{where } E_N^{\mathrm{HF}} := \inf_{\substack{\psi \text{ is Slater} \\ \text{determinant}}} \langle \psi, H_N \psi \rangle \ .$$

Achieved restriction to simplest fermionic states: antisymmetrized tensor products,

$$\psi_N = \text{Slater determinant} = \bigwedge_{j=1}^{N} f_j \ , \qquad f_j \in L^2(\mathbb{T}^3) \ .$$

# Hartree–Fock Theory = Restriction to Slater Determinants

Multiply the entire Hamiltonian by $\times\hbar^2$, with $\hbar := N^{-1/3}$:

$$H_N = \sum_{i=1}^{N} \left(-\hbar^2 \Delta_i\right) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V(x_i - x_j) \; .$$

Convergence to the Hartree–Fock energy [Bach '92, Graf–Solovej '94]:

$$\left| E_N - E_N^{\mathrm{HF}} \right| = o(N) \, , \qquad \text{where } E_N^{\mathrm{HF}} := \inf_{\substack{\psi \text{ is Slater} \\ \text{determinant}}} \langle \psi, H_N \psi \rangle \; .$$

Achieved restriction to simplest fermionic states: antisymmetrized tensor products,

$$\psi_N = \text{Slater determinant} = \bigwedge_{j=1}^{N} f_j \; , \qquad f_j \in L^2(\mathbb{T}^3) \; .$$

Moreover: Consider a Slater determinant and evolve it in time — it stays close to a Slater determinant, but with orbitals $f_{j,t}$ evolved under the time–dependent Hartree–Fock equation [B–Porta–Schlein '14]. This is optimal [B–Sok–Solovej '18].

# The Almost–Optimal Slater Determinant

Introduce the Slater determinant of $N$ plane waves

$$\Psi_N := \bigwedge_{k \in \mathcal{B}_F} f_k \,, \qquad \mathcal{B}_F = \text{Fermi ball} := \left\{ k \in \mathbb{Z}^3 \mid |k| \leq N^{1/3} \left( \frac{3}{4\pi} \right)^{1/3} \right\}.$$

For the translation–invariant situation on the torus [Gontier–Hainzl–Lewin '18]:

$$E_N^{\mathrm{pw}} := \langle \Psi_N, H_N \Psi_N \rangle = E_N^{\mathrm{HF}} + \mathcal{O}(e^{-N^{1/3}}) \,.$$

To summarize:

$$E_N < E_N^{\mathrm{HF}} \simeq E_N^{\mathrm{pw}} \,.$$

# The Almost–Optimal Slater Determinant

Introduce the Slater determinant of $N$ plane waves

$$\Psi_N := \bigwedge_{k \in \mathcal{B}_F} f_k \,, \qquad \mathcal{B}_F = \text{Fermi ball} := \left\{ k \in \mathbb{Z}^3 \mid |k| \leq N^{1/3} \left(\frac{3}{4\pi}\right)^{1/3} \right\}.$$

For the translation–invariant situation on the torus [Gontier–Hainzl–Lewin '18]:

$$E_N^{\text{pw}} := \langle \Psi_N, H_N \Psi_N \rangle = E_N^{\text{HF}} + \mathcal{O}(e^{-N^{1/3}}) \,.$$

To summarize:

$$E_N < E_N^{\text{HF}} \simeq E_N^{\text{pw}} \,.$$

[Wigner '34]: How to compute the correlation energy $E_N - E_N^{\text{HF}}$ (or $E_N - E_N^{\text{pw}}$)?

Do better than antisymm. tensor products, include non–trivial quantum correlations!

# Describing Correlations by Bosonization

# Separating the Slater Determinant

Hamiltonian in momentum representation, written with anti-commuting operators:

$$H_N := \hbar^2 \sum_{k\in\mathbb{Z}^3} |k|^2 a_k^* a_k + \frac{1}{N} \sum_{q,s,k\in\mathbb{Z}^3} \hat{V}(k) a_{q+k}^* a_{s-k}^* a_s a_q \,, \qquad \hbar = N^{-1/3} \,.$$

Define the unitary map $R$ ("particle–hole transformation") on fermionic Fock space by

$$R\,|0\rangle := \Psi_N \,, \qquad\qquad R\, a_k^*\, R^* := \begin{cases} a_k^* & k \in \mathcal{B}_F^c \\ a_k & k \in \mathcal{B}_F \end{cases}$$

## Separating the Slater Determinant

Hamiltonian in momentum representation, written with anti-commuting operators:

$$H_N := \hbar^2 \sum_{k\in\mathbb{Z}^3} |k|^2 a_k^* a_k + \frac{1}{N} \sum_{q,s,k\in\mathbb{Z}^3} \hat{V}(k) a_{q+k}^* a_{s-k}^* a_s a_q \,, \qquad \hbar = N^{-1/3} \,.$$

Define the unitary map $R$ ("particle–hole transformation") on fermionic Fock space by

$$R\,|0\rangle := \Psi_N \,, \qquad\qquad R\, a_k^*\, R^* := \begin{cases} a_k^* & k \in \mathcal{B}_F^c \\ a_k & k \in \mathcal{B}_F \end{cases}$$

Write $\tilde{\Psi}_N = R\xi$, expand $R^* H_N R$, normal–order: with $e(p) := \left|\hbar^2 |p|^2 - (3/4\pi)^{2/3}\right|$ get

$$\langle \tilde{\Psi}_N, H_N \tilde{\Psi}_N \rangle = E_N^{\mathrm{HF}} + \langle \xi, \Big[ \underbrace{\sum_{p\in\mathcal{B}_F^c} e(p) a_p^* a_p + \sum_{h\in\mathcal{B}_F} e(h) a_h^* a_h}_{=:\ H_{\mathrm{kin}}} + \underbrace{Q}_{\text{quartic in operators } a^*,\, a} \Big] \xi \rangle$$

Slater determinant corresponds to $\xi = |\Omega\rangle$: in particular $(H_{\mathrm{kin}} + Q)\,|0\rangle = 0$.

## Collective Particle–Hole Pairs

**Key observation:** if we introduce collective pair creation operators

$$b_k^* := \sum_{\substack{p \in \mathcal{B}_F^c \\ h \in \mathcal{B}_F}} \delta_{p-h,k} a_p^* a_h^*$$

$p$ "particle" outside the Fermi ball

$h$ "hole" inside the Fermi ball

then

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \Big( 2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \Big) + Q_{\text{non-paired}} \,.$$

This is convenient because the $b_k^*$ and $b_k$ have approximately bosonic commutators:

$$[b_k^*, b_l^*] = 0 \quad , \qquad [b_l, b_k^*] = \delta_{k,l} n_k^2 + \cancel{\mathcal{E}(k,l)} \,.$$

# Collective Particle–Hole Pairs

**Key observation:** if we introduce collective pair creation operators

$$b_k^* := \sum_{\substack{p\in\mathcal{B}_F^c\\ h\in\mathcal{B}_F}} \delta_{p-h,k} a_p^* a_h^*$$

$p$ "particle" outside the Fermi ball

$h$ "hole" inside the Fermi ball

then

$$Q = \frac{1}{N}\sum_{k\in\mathbb{Z}^3} \hat{V}(k)\Big(2b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k\Big) + Q_{\text{non-paired}}\,.$$

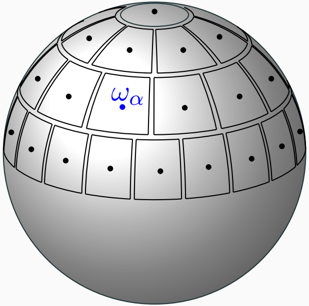
This is convenient because the $b_k^*$ and $b_k$ have approximately bosonic commutators:

$$[b_k^*, b_l^*] = 0 \quad , \qquad [b_l, b_k^*] = \delta_{k,l} n_k^2 + \cancel{\mathcal{E}(k,l)}\,.$$

But how to express $H_{\text{kin}}$ through pair operators?

# Linearizing the Kinetic Energy Locally in Patches

Fermi ball $\mathcal{B}_F$




[Benfatto–Gallavotti '90]

[Haldane '94]

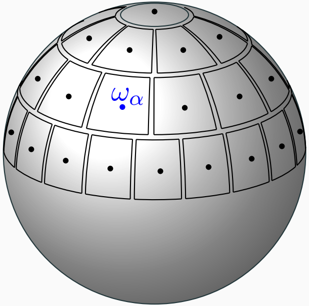
Localize to $M = M(N)$ patches near the Fermi surface,

$$b^*_{\alpha,k} := \frac{1}{n_{\alpha,k}} \sum_{\substack{p \in \mathcal{B}_F^c \cap B_\alpha \\ h \in \mathcal{B}_F \cap B_\alpha}} \delta_{p-h,k} a^*_p a^*_h$$

with $n_{\alpha,k}$ such that $\| b^*_{\alpha,k} |0\rangle \| = 1$.

# Linearizing the Kinetic Energy Locally in Patches

Fermi ball $\mathcal{B}_F$




[Benfatto–Gallavotti '90]

[Haldane '94]

Localize to $M = M(N)$ patches near the Fermi surface,

$$b^*_{\alpha,k} := \frac{1}{n_{\alpha,k}} \sum_{\substack{p \in \mathcal{B}_F^c \cap B_\alpha \\ h \in \mathcal{B}_F \cap B_\alpha}} \delta_{p-h,k} a^*_p a^*_h$$

with $n_{\alpha,k}$ such that $\| b^*_{\alpha,k} |0\rangle \| = 1$.

Linearize kinetic energy around patch center $\omega_\alpha$:

$$H_{\text{kin}} b^*_{\alpha,k} |0\rangle \simeq 2\hbar |k \cdot \hat{\omega}_\alpha| b^*_{\alpha,k} |0\rangle$$

Approximate as if $b^*_{\alpha,k}$ was a harmonic oscillator mode:

$$H_{\text{kin}} \simeq \sum_{k \in \mathbb{Z}^3} \sum_{\alpha=1}^{M} 2\hbar u_\alpha(k)^2 b^*_{\alpha,k} b_{\alpha,k} , \quad u_\alpha(k)^2 := |k \cdot \hat{\omega}_\alpha|$$

(c. f., [Lieb–Mattis '65] for the 1D Luttinger model).

# Quadratic Effective Hamiltonian

Recall

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left( 2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \right) + Q_{\text{non-paired}}$$

Decompose

$$b_k^* = \sum_{\alpha} n_{\alpha,k} b_{\alpha,k}^* + \text{lower order} \; .$$

# Quadratic Effective Hamiltonian

Recall

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left( 2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \right) + Q_{\text{non-paired}}$$

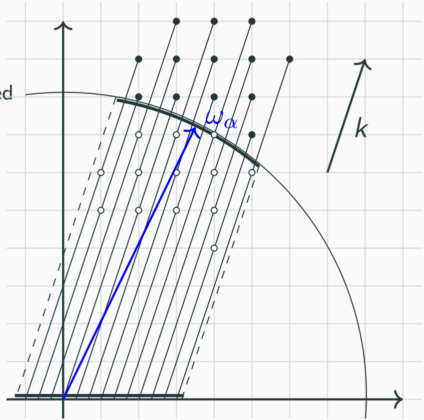
Decompose

$$b_k^* = \sum_\alpha n_{\alpha,k} b_{\alpha,k}^* + \text{lower order} \; .$$

Normalization:

$$n_{\alpha,k}^2 = \#\text{p-h pairs in patch } B_\alpha \text{ with momentum } k$$

$$\simeq \frac{4\pi N^{2/3}}{M} |k \cdot \hat{\omega}_\alpha| = \frac{4\pi N^{2/3}}{M} u_\alpha(k)^2 \; .$$



# Quadratic Effective Hamiltonian

Recall

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left( 2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \right) + Q_{\text{non-paired}}$$

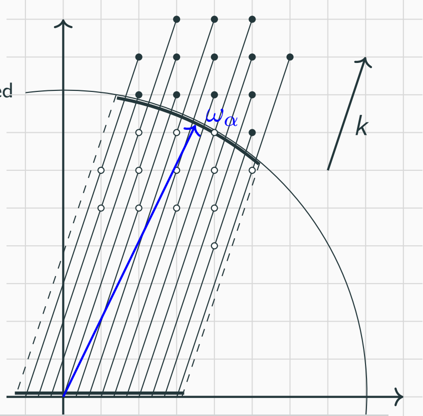
Decompose

$$b_k^* = \sum_\alpha n_{\alpha,k} b_{\alpha,k}^* + \text{lower order} .$$

Normalization:

$$n_{\alpha,k}^2 = \#\text{p-h pairs in patch } B_\alpha \text{ with momentum } k$$

$$\simeq \frac{4\pi N^{2/3}}{M} |k \cdot \hat{\omega}_\alpha| = \frac{4\pi N^{2/3}}{M} u_\alpha(k)^2 .$$




## Effective Bosonic Hamiltonian

$$H^{\text{eff}} = \hbar \sum_{k \in \mathbb{Z}^3} \left[ \sum_\alpha u_\alpha(k)^2 b_{\alpha,k}^* b_{\alpha,k} + \frac{\hat{V}(k)}{M} \sum_{\alpha,\beta} \left( u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,k} + u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,-k}^* + \text{h.c.} \right) \right]$$

# Diagonalization of the Bosonic Hamiltonian

We can write
$$H^{\text{eff}} = \hbar \sum_{k \in \mathbb{Z}^3} \left[ h^{\text{eff}}(k) - \frac{1}{2} \operatorname{tr}(D(k) + W(k)) \right]$$

where
$$h^{\text{eff}} = \frac{1}{2} \begin{pmatrix} (b^*)^T & b^T \end{pmatrix} \begin{pmatrix} D + W & \widetilde{W} \\ \widetilde{W} & D + W \end{pmatrix} \begin{pmatrix} b \\ b^* \end{pmatrix}, \qquad b = \begin{pmatrix} \vdots \\ b_\alpha \\ \vdots \end{pmatrix},$$

$$D = \begin{pmatrix} \operatorname{diag}(u_\alpha^2) & 0 \\ 0 & \operatorname{diag}(u_\alpha^2) \end{pmatrix}, \quad W = \hat{V} \begin{pmatrix} |u\rangle\langle u| & 0 \\ 0 & |u\rangle\langle u| \end{pmatrix}, \quad \widetilde{W} = \hat{V} \begin{pmatrix} 0 & |u\rangle\langle u| \\ |u\rangle\langle u| & 0 \end{pmatrix}.$$

The model is solved (spectrum computed) if we can find linear combinations of $b$– and $b^*$–operators, called $\tilde{b}$, with the same CCR such that
$$h^{\text{eff}} = \sum_{\gamma=1}^{M} e_\gamma \left( \tilde{b}_\gamma^* \tilde{b}_\gamma + \frac{1}{2} \right), \quad e_\gamma \in \mathbb{R}.$$

# Bogoliubov Transformation

These are Bogoliubov transformations (or complexified symplectic transformations).

We achieve diagonalization up to a one–particle unitary [B '19] (which is sufficient):

$$h^{\text{eff}} = \sum_{\gamma,\delta=1}^{M} E_{\gamma,\delta}\, \tilde{b}^*_\gamma\, \tilde{b}_\delta + \frac{1}{2}\operatorname{tr} E\,, \qquad E \simeq \sqrt{\operatorname{diag}(u_\alpha^2) + \hat{V}|u\rangle\langle u|} \geq 0\,.$$

In the limit of large number of patches, $M \to \infty$, the correlation energy becomes

$$\hbar \sum_{k\in\mathbb{Z}^3} \frac{1}{2}\operatorname{tr}\left(E(k) - D(k) - W(k)\right)$$

$$\to E_N^{\text{RPA}} := \hbar \sum_{k\in\mathbb{Z}^3} |k| \left[\int_0^\infty \log\left(1 + \hat{V}(k)\left(1 - \lambda \arctan \lambda^{-1}\right)\right) \mathrm{d}\lambda - \frac{1}{4}\hat{V}(k)\right]\,.$$

Predicted by partial resumm. of pert. theory [Macke '50, Gell-Mann–Brueckner '57].

# Rigorous Result: Correlation Energy

## Leading Order of the Correlation Energy

**Theorem:** [B–Nam–Porta–Schlein–Seiringer '19, '20]

Let $\hat{V}(k)$ be non-negative and compactly supported with sufficiently small $\|\hat{V}\|_{\ell^\infty}$. Then as number of particles $N \to \infty$ we have

$$E_N = E_N^{\mathrm{HF}} + E_N^{\mathrm{RPA}} + \mathcal{O}(\hbar^{1+\frac{1}{48}}) \qquad (\hbar = N^{-1/3}) \; .$$

# Leading Order of the Correlation Energy

**Theorem:** [B–Nam–Porta–Schlein–Seiringer '19, '20]

Let $\hat{V}(k)$ be non-negative and compactly supported with sufficiently small $\|\hat{V}\|_{\ell^\infty}$. Then as number of particles $N \to \infty$ we have

$$E_N = E_N^{\mathrm{HF}} + E_N^{\mathrm{RPA}} + \mathcal{O}(\hbar^{1+\frac{1}{48}}) \qquad (\hbar = N^{-1/3}) .$$

The upper bound follows from a computation with a trial state (variational principle).

Proof of the lower bound is the rest of the talk.

*Remarks:* [Hainzl–Porta–Rexze '19] obtained a rigorous lower bound to second order in $\hat{V}$,

$$E_N \geq E^{\mathrm{HF}} - \hbar\frac{\pi}{2}(1 - \log 2) \sum_{k \in \mathbb{Z}^3} |k| \hat{V}(k)^2 + \mathcal{O}(\hat{V}^3) .$$

# Proof of the Lower Bound

# Explicit Diagonalization of the Hamiltonian

The previously introduced Bogoliubov transformation has an explicit formula:

$$T = \exp\left( \sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} K(k)_{\alpha,\beta} b^*_{\alpha,k} b^*_{\beta,-k} - \text{h.c.} \right) , \quad K(k) = \log|S_1|$$

and thus, in the exactly bosonic picture,

$$T\, h^{\text{eff}}\, T^* = \sum_{\gamma,\delta=1}^{M} E_{\gamma,\delta}\, b^*_\gamma\, b_\delta + \frac{1}{2} \operatorname{tr} E \quad \geq \frac{1}{2} \operatorname{tr} E \ .$$

Makes sense even if the $b^*$–operators are actually pairs of fermionic operators.

To control: gapless excitations, non–bosonizable terms, errors in approximate CCR.

# Strategy (1/2)

1. A–priori estimate: $\|b(k)\psi\| \leq CN^{1/2}\|H_{\text{kin}}^{1/2}\psi\|$ [Hainzl–Porta–Rexze '19]. Thus
$$H_{\text{kin}} \leq C(H_{\text{kin}} + Q) \leq \hbar \, .$$
2. Bound on the number operator: $|\{\text{lattice points on the sphere}\}| \leq C_\epsilon N^{1/3+\epsilon}$, thus
$$\mathcal{N} := \sum_{i\in\mathbb{Z}^3} a_i^* a_i \leq \sum_{e(i)\leq N^{-\theta}} 1 + \sum_{e(i)>N^{-\theta}} a_i^* a_i \overset{\theta=2/3}{\leq} CN^{1/3+\epsilon} + N^{2/3}H_{\text{kin}} = \mathcal{O}(N^{1/3}) \, .$$
3. IMS localization in Fock space to control also powers of $\mathcal{N}$.
4. Removing gapless pair excitations (momentum approximately parallel to Fermi surface) and corridors from Hamiltonian using $H_{\text{kin}}$–, $\mathcal{N}$–bounds.
5. Approximate CCR:
$$[b_{\alpha,k}, b_{\beta,l}^*] = \delta_{\alpha,\beta}\left(\delta_{k,l} + \mathcal{E}_\alpha(k,l)\right) \, , \qquad \mathcal{E}_\alpha(k,l) \text{ bounded by } \mathcal{N} \, .$$
6. $T b_{\alpha,k} T^* = \sum_\gamma \cosh(K)_{\alpha,\gamma} b_{\gamma,k} + \sum_\gamma \sinh(K)_{\alpha,\gamma} b_{\gamma,k}^* + \mathfrak{E}_{\alpha,k}$ (almost bosonic BT).

# Strategy (2/2)

7. Major improvement I: boson bounds avoiding the uncontrollable energy gap

   $\|b_{\alpha,k}\psi\| \leq \|\mathcal{N}_\delta^{1/2}\psi\|$ where $\mathcal{N}_\delta := \sum_{e(i)>\frac{1}{4}N^{-1/3-\delta}} a_i^* a_i$ , $\delta$ to be optimized.

8. Major improvement II: strong control on the Bogoliubov kernel $K(k)$

$$|K(k)_{\alpha,\beta}| \leq \frac{C}{M} \min\left\{\frac{u_\alpha(k)}{u_\beta(k)}, \frac{u_\beta(k)}{u_\alpha(k)}\right\} . \tag{1}$$

9. Major improvement III: strong bound on linearization of kinetic energy permits $M \gg N^{2\delta}$ instead of $M \gg N^{1/3}$.

10. Major improvement IV: non–bosonizable terms

    partial transformation & completing the square $\Rightarrow Q_{\text{non–pair}} \geq -\|\hat{V}\|_{\ell^\infty} H_{\text{kin}}$ .

11. $H_{\text{kin}} + Q = H_{\text{kin}} - H^{\text{B}}_{\text{kin}} + H^{\text{B}}_{\text{kin}} + Q$. We have $T\left(H_{\text{kin}} - H^{\text{B}}_{\text{kin}}\right) T^* \simeq \left(H_{\text{kin}} - H^{\text{B}}_{\text{kin}}\right)$,

    $H^{\text{B}}_{\text{kin}} + Q = \sum_{\gamma,\delta=1}^{M} E_{\gamma,\delta}\, \tilde{b}^*_\gamma\, \tilde{b}_\delta + \frac{1}{2}\operatorname{tr} E \geq H^{\text{B}}_{\text{kin}} - \|\hat{V}\|_{\ell^\infty} H_{\text{kin}} + \frac{1}{2}\operatorname{tr} E$ using (1).

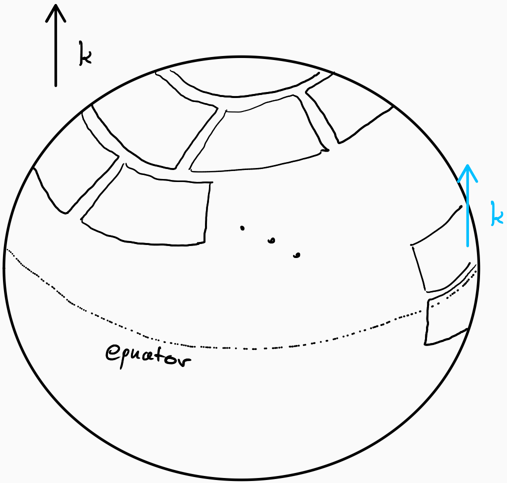
# Kinetic bound [Hainzl–Porta–Rexze '19]

*Proof of the kinetic bound.*

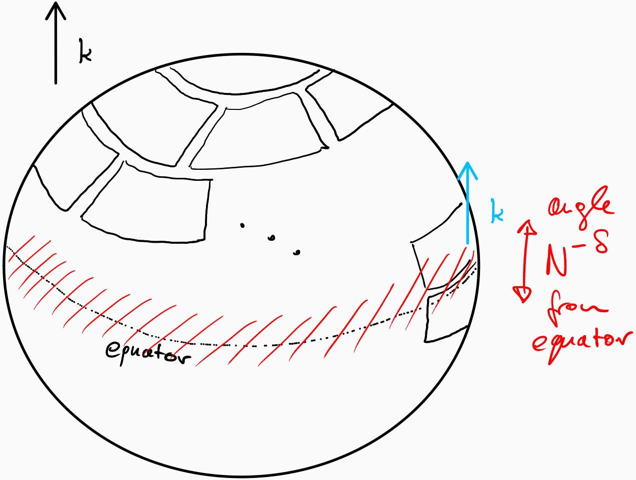
$$\begin{aligned}
\|b(k)\psi\| &\leq \sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}\|a_h a_p\psi\| = \sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}\frac{1}{\sqrt{e(p)+e(h)}}\sqrt{e(p)+e(h)}\|a_h a_p\psi\| \\
&\leq \Bigg[\sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}\frac{1}{e(p)+e(h)}\Bigg]^{1/2}\Bigg[\sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}(e(p)+e(h))\|a_h a_p\psi\|^2\Bigg]^{1/2} \\
&\leq \Bigg[\sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}\frac{1}{e(p)+e(h)}\Bigg]^{1/2}\Bigg[\sum_{\substack{p\in B_\mathrm{F}^c\\ h\in B_\mathrm{F}}} \delta_{p-h,k}\Big(e(p)\|a_p\psi\|^2+e(h)\|a_h\psi\|^2\Big)\Bigg]^{1/2} \\
&\leq CN^{1/2}\|H_\mathrm{kin}^{1/2}\psi\|\;.
\end{aligned}$$

The sum is singular near the Fermi surface – estimate by $CN^{1/2}$ requires use of number theoretic results about counting of lattice points! $\square$

# Removing Gapless Excitations



# Removing Gapless Excitations




Introduce a cut–off: instead of $b_k^* = \sum_{\alpha=1^M} n_{\alpha,k} b^*_{\alpha,k}$ consider

$$b_k^* \simeq \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha,k} b^*_{\alpha,k} \;,$$

where

$$\mathcal{I}_k^+ := \Big\{ \alpha \in \{1, 2, \ldots, M\} : \; k \cdot \hat{\omega}_\alpha > N^{-\delta} \Big\} \;.$$

Difference can be controlled by

$$(\ldots) \leq C N^{1/2 - \delta/2} \| H_{\text{kin}}^{1/2} \psi \| \;.$$

# Boson bounds avoiding the uncontrollable energy gap

*Proof of* $\|b_{\alpha,k}\psi\| \leq \|\mathcal{N}_\delta^{1/2}\psi\|$. Recall $\mathcal{N}_\delta := \sum_{e(i)>\frac{1}{4}N^{-1/3-\delta}} a_i^* a_i$, $\delta$ as in the cut–off.

$$\|b_{\alpha,k}\psi\| \leq \frac{1}{n_\alpha(k)} \sum_{\substack{p\in B_F^c \\ h\in B_F}} \delta_{p-h,k} \|a_h a_p \psi\|$$

$$\leq \frac{1}{n_\alpha(k)} \sum_{\substack{p\in B_F^c \\ h\in B_F}} \delta_{p-h,k} \min\{\|a_p\psi\|, \|a_h\psi\|\} \, .$$

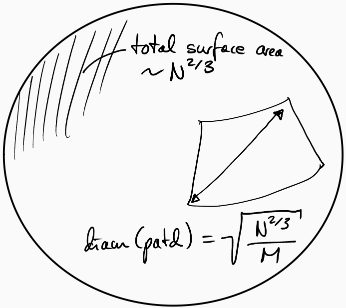
Due to $\alpha \in \mathcal{I}_k^+$ we have $k \cdot \hat{\omega}_\alpha > N^{-\delta}$, thus

$$e(p) + e(h) = \hbar^2 p^2 - \hbar^2 h^2 = \hbar^2 \left(p^2 - (p-k)^2\right) = \hbar^2 \left(2p\cdot k - k^2\right)$$

$$\simeq \hbar^2 2\omega_\alpha \cdot k = 2(\tfrac{3}{4\pi})^{1/3}\hbar k \cdot \hat{\omega}_\alpha \geq \frac{1}{2} N^{-1/3-\delta} \, .$$

Thus $e(p) > \frac{1}{4}N^{-1/3-\delta}$ or $e(h) > \frac{1}{4}N^{-1/3-\delta}$. “Lower bound on excitation energy”. $\square$

# Linearization of the Kinetic Energy




*Instead of linearizing $H_{\mathrm{kin}}$, only linearize its commutator,* retaining info about "away from the energy gap" as last bound.

$$[H_{\mathrm{kin}}, b^*_{\alpha,k}] = \frac{1}{n_\alpha(k)} \sum_{\substack{p \in B_F^c \cap B_\alpha \\ h \in B_F \cap B_\alpha}} (e(p) + e(h))\, \delta_{p-h,k} a^*_p a^*_h$$

$$= 2\hbar(\tfrac{3}{4\pi})^{1/3} k \cdot \hat{\omega}_\alpha b^*_{\alpha,k} + \text{error}$$

$$\text{error} \sim \left(e(p) + e(p-k) - 2\hbar(\tfrac{3}{4\pi})^{1/3} k \cdot \hat{\omega}_\alpha\right) = \hbar^2 \left(2k \cdot (p - \omega_\alpha) - k^2\right)$$

$$\leq \hbar^2 C |p - \omega_\alpha| \leq \hbar^2 C \,\mathrm{diam}(B_\alpha) \leq \hbar^2 \frac{N^{1/3}}{\sqrt{M}} = \hbar \mathcal{O}(M^{-1/2})\ .$$

Error bounded by $\hbar \mathcal{N}_\delta^{1/2} M^{-1/2} \ll \hbar$. Major improvement: $M \gg N^\delta$ is sufficient. $\square$

# Non–Bosonizable Terms

*Non–bosonizable terms.* Positive non–pair term + coupling to pairs

$$Q_{\text{non–pair}} = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left( D^*(k)D(k) + D^*(k)b(k) + b(k)D^*(k) \right) .$$

Of course

$$D^*(k)b(k) \geq -\frac{1}{2}D^*(k)D(k) - \frac{1}{2}b^*(k)b(k) \geq -\frac{1}{2}D^*(k)D(k) - \frac{1}{2}H_{\text{kin}} .$$

To use this bound we need to first transform the $b$–operators with the Bogoliubov transformation, and afterward use a similar bound. Only possible because

$$|K(k)_{\alpha,\beta}| \leq \tfrac{C}{M} \min\left\{ \tfrac{u_\alpha(k)}{u_\beta(k)}, \tfrac{u_\beta(k)}{u_\alpha(k)} \right\} \quad \Rightarrow \quad \sum_{\alpha} K_{\alpha,\beta} n_\alpha \leq C n_\beta .$$

allows us to remove the $\cosh(K)$ and $\sinh(K)$ from the transformed expression. $\square$

QED