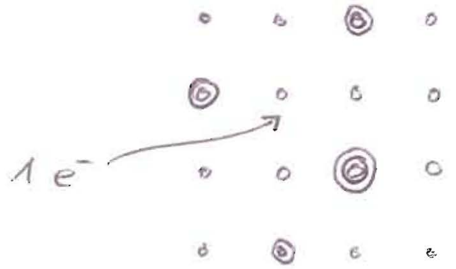
Anderson Localization (Graf, J. Stat. Phys., Vol. 75, 1994, p. 337 – 346)

Let us start by looking at the physical picture:

- lattice built of atoms
- alloy ⇝ different kinds of atoms randomly distributed
- 1 electron on the lattice: will it stay localized or will it's wave function extend?



Electrically conducting or isolator?

2 Methods: Multi-scale and Aizenman for Lattice.

Hilbert space: $\ell^2(\mathbb{Z}^d)$, $d \in \mathbb{N}$. (d arbitrary)

Hamiltonian: $h_\omega = -\Delta + v_\omega$

where $(\Delta\psi)(x) = \sum_{|e|=1} \psi(x+e)$ (discrete Laplacian)

and $(v_\omega\psi)(x) = v_x\,\psi(x)$, (random potential)

$\omega = \{v_x\}_{x\in\mathbb{Z}^d}$ i.i.d. r.v. (independent identically distributed)

Assumption: single-site prob. distr. has a (compactly supported) density $\rho \in L^1(\mathbb{R})$, $\|\rho\|_1 = 1$, w.r.t. Lebesgue.

i.e. $\omega \in \Omega = \bigtimes_{x\in\mathbb{Z}^d} \mathbb{R}$,

$$dP(\omega) = \prod_{x\in\mathbb{Z}^d} \rho(v_x)\,dv_x .$$

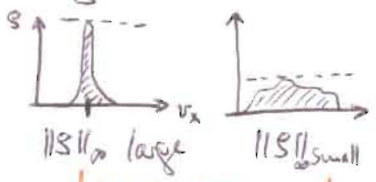
Remark: $\|h_\omega\psi\| \le \|\sum_{|e|=1}\psi(\cdot+e)\| + \|v_\omega\psi\|$

$\le 2d\,\|\psi\| + \sup|\operatorname{supp}\rho|\,\|\psi\|.$

Thus $\sigma(h_\omega) \subset [\inf(\operatorname{supp}\rho) - 2d, \sup(\operatorname{supp}\rho) + 2d]$. ∎

(We'll need this later on.)

Thm.: If $\|\rho\|_\infty$ small enough, $\longleftarrow$ large disorder
then almost surely
h_ω has only pure point spectrum,
i.e. localization.



$\hookrightarrow$ only one v_x dominates

many v_x possible

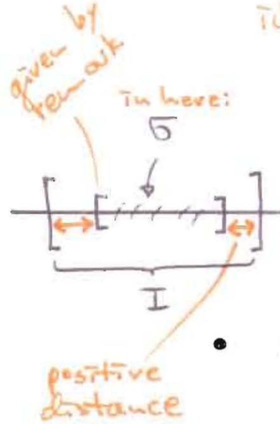
Why almost surely: With $P=0$ we could get $v_x = 0 \ \forall x \in \mathbb{Z}^d$, i.e. free electron, no localization.

Proof: RAGE:

$$\|E_c \psi\|^2 = \lim_{R\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T ds \, \| P_{|x|\geq R}\, e^{-i h_\omega s} \psi \|^2$$

standard identity

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int dE \, \| P_{|x|\geq R} (h_\omega - E - i\varepsilon)^{-1} \psi\|^2$$



Choose compact $I \subset \mathbb{R}$ s.th. $\sigma(h_\omega) \subset I$. (By remark, I does not depend on ω.)

We want to show: r.h.s. $\to 0$

- $\varepsilon \int_{\mathbb{R}\setminus I} dE \, \| P_{|x|\geq R} (h_\omega - E - i\varepsilon)^{-1} \psi \|^2$

$$\leq \varepsilon \int_{\mathbb{R}\setminus I} dE \, \| (h_\omega - E - i\varepsilon)^{-1} \psi\|^2$$

$$\leq \varepsilon \|\psi\|^2 \sup_{\lambda\in\sigma(h_\omega)} \int_{\mathbb{R}\setminus I} dE \, |\lambda - E - i\varepsilon|^{-2}$$

because σ and I^c have distance > 0.

$$\leq \varepsilon \|\psi\|^2 \sup_{\lambda\in\sigma(h_\omega)} \int_{\mathbb{R}\setminus I} dE \, |\lambda - E|^{-2} \longrightarrow 0 \quad (\varepsilon \to 0).$$

- w.l.o.g. $\psi = \delta_0$. (δ_x form basis, translation trivial)

$$\|E_c \delta_0\|^2 = \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_I dE \, \| P_{|x|\geq R} (h_\omega - E - i\varepsilon)^{-1} \delta_0 \|^2$$

with $P_{|x|\geq R} = \sum_{x:\,|x|\geq R} (\delta_x, \cdot)\, \delta_x$

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_I dE \sum_{x,y} |(\delta_x, (h_\omega - E - i\varepsilon)^{-1} \delta_0)|^2 \, (\delta_x, \delta_y)$$

by Def. the Green's function, matrix elem. of resolvent.

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_I dE \sum_{x:\,|x|\geq R} |G_\omega(x, 0; E + i\varepsilon)|^2.$$

Exchange Expectation with lim inf if we write \lep.

By Fatou:

$$\mathbb{E}_\omega \, \| E_c \delta_0 \|^2 \le \varliminf_{R\to\infty} \varliminf_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_I dE \sum_{x:|x|\ge R} \mathbb{E}_\omega \, |G_\omega(x,0;E+i\varepsilon)|^2 .$$

Lemma C: (Exponential bound on Green's function)

Let $\|\rho\|_\infty$ small enough and ρ of compact support. Then $\exists\, C, m>0$ s.th.

$$|\mathrm{Im}\, z| \; \mathbb{E}_\omega \, |G_\omega(x,y;z)|^2 \le C e^{-m|x-y|}$$

$\forall z \in \mathbb{C}\setminus\mathbb{R}$, $x,y\in\mathbb{Z}^d$.

// do not erase

Thus

$$\mathbb{E}_\omega \, \|E_c \delta_0\|^2 \le \varliminf_{R\to\infty} \varliminf_{\varepsilon\downarrow 0} \frac{\overset{\mathrm{Im}}{\cancel{\varepsilon}}}{\pi} \int_I dE \sum_{x:|x|\ge R} C e^{-m|x|}$$

$$\le \frac{C|I|}{\pi} \varliminf_{R\to\infty} \sum_{x:|x|\ge R} e^{-m|x|} = 0 .$$

But if the expectation of a non-negative function is zero, so is the function up to a set of measure zero.

$\Rightarrow \|E_c \delta_0\| = 0$ almost surely.

Similarly $\|E_c \delta_x\| = 0 \;\; \forall x \in \mathbb{Z}^d$, so $E_c = 0$. almost surely

∎

We will now prove two Lemmata, which will give us a proof of Lemma C.

Lemma A: Let $0<s<1$. Then $\exists\, C>0$ s.th. $\mathbb{E}_\omega \, |G_\omega(x,y;z)|^s \le C \|\rho\|_\infty^s$ $\forall z\in\mathbb{C}\setminus\mathbb{R}$, $x,y\in\mathbb{Z}^d$.

Notice $s<1$, to be extended in Lm. 3.

Proof:

Treat dependence on V_x, V_y as rank-2 perturbation. Assume $x \neq y$. ($x=y$ similar but easier)

$= (\delta_x, \cdot)\delta_x$ projection on site x

Write $h_\omega = h_{\hat\omega} + v_x P_x + v_y P_y$,
with $\hat\omega$ obtained from ω by setting
$v_x = v_y = 0$.

Set $P := P_x + P_y$. Resolvent identities $\rightsquigarrow$

(Krein) $\begin{cases} P(h_\omega - z)^{-1}P = (A + v_x P_x + v_y P_y)^{-1} : \operatorname{ran} P \to \operatorname{ran} P \\ \text{with } A = (P(h_{\hat\omega} - z)^{-1} P)^{-1} \text{ \underline{indep. of v_x and v_y.}} \end{cases}$

(One has to think about existence of the inverse)

In matrix notation w.r.t. (δ_x, δ_y):

$$A = \begin{pmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{pmatrix}, \quad \operatorname{Im} A = \frac{A - A^*}{2i} = \begin{pmatrix} \operatorname{Im} a_{xx} & \frac{1}{2i}(a_{xy} - \overline{a_{yx}}) \\ \frac{1}{2i}(a_{yx} - \overline{a_{xy}}) & \operatorname{Im} a_{yy} \end{pmatrix}.$$

Thus

$$\begin{aligned} G_\omega(x,y;z) &= \langle \delta_x, (h_\omega - z)^{-1} \delta_y \rangle \\ &= \langle \delta_x, P(h_\omega - z)^{-1} P \delta_y \rangle \\ &\overset{\text{(Krein)}}{=} (1,0) \begin{pmatrix} a_{xx} + v_x & a_{xy} \\ a_{yx} & a_{yy} + v_y \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= - \frac{a_{xy}}{(v_x + a_{xx})(v_y + a_{yy}) - a_{xy} a_{yx}} \end{aligned} \qquad \text{(I)}$$

to be needed later on

Elementary but lengthy estimates $\rightsquigarrow$

$$|G_\omega(x,y;z)|^s \le C \left(\cancel{\frac{1}{|v_x - \underbrace{\beta_1(a_{ij}, v_y)}_{\in \mathbb{R}}|^s}} + \frac{1}{|v_y - \beta_2(a_{ij}, v_x)|^s} \right).$$

w.l.o.g. (let's leave away one of this terms, they are similar)

Thus

$$\mathbb{E}_\omega |G_\omega(x,y;z)|^s$$

$$= \underbrace{\int_{\text{„}\mathbb{Z}^d \setminus \{x,y\}\text{"}} dP(\hat\omega) \int dv_x\, \rho(v_x)}_{=1} \underbrace{\int dv_y\, \rho(v_y) \frac{C}{|v_y - \beta(a_{ij}, v_x)|^s}}_{\le C_s \underbrace{\|\rho\|_1^{1-s}}_{=1} \|\rho\|_\infty^s, \text{ \underline{indep. of β}}}$$

$$\le C \|\rho\|_\infty^s. \qquad \blacksquare$$

Will now show that bound indep. of x and y from Lm. A implies an exponential bound.

Lemma B: Let $\|g\|_\infty$ small enough and $0 < s < 1$. Then $\exists C, m > 0$ s.th.

$$\mathbb{E}_\omega |G_\omega(x,y;z)|^s \leq C e^{-m|x-y|}$$

$\forall z \in \mathbb{C} \setminus \mathbb{R}$, $x, y \in \mathbb{Z}^d$.

Proof: According to (I): $G_\omega(x,y;z) = \dfrac{\alpha}{v_y - \beta}$, (II)

with α, β independent of v_y.

(but depending on v_i for $i \neq y$)

Now for $y \neq x$:

$$0 = \langle \delta_x, \mathbb{1} \delta_y \rangle = \langle \delta_x, (h_\omega - z)^{-1}(h_\omega - z)\delta_y \rangle$$

$$= \langle \delta_x, (h_\omega - z)^{-1} h_\omega \delta_y \rangle - z \underbrace{\langle \delta_x, (h_\omega - z)^{-1} \delta_y \rangle}_{\text{this we can already identify} \to = G_\omega(x,y;z)} .$$

Let's see what we can do for the first term:

We have $(h_\omega \delta_y)(x) = -\sum_{|e|=1} \delta_y(x+e) + (v_\omega \delta_y)(x)$,

so $h_\omega \delta_y = -\sum_{|e|=1} \delta_{y+e} + v_y \delta_y$,

thus

$$0 = -\sum_{|e|=1} \langle \delta_x, (h_\omega - z)^{-1} \delta_{y+e} \rangle + \langle \delta_x, (h_\omega - z)^{-1} \delta_y \rangle v_y - G_\omega(x,y;z) z$$

$$= -\sum_{|e|=1} G_\omega(x, y+e; z) + (v_y - z) G_\omega(x,y;z)$$

now use that the root is concave:

$$\sum_{|e|=1} |G_\omega(x, y+e; z)|^s \geq |v_y - z|^s |G_\omega(x,y;z)|^s .$$

Taking the expectation:

$$\mathbb{E}_\omega \sum_{|e|=1} |G_\omega(x, y+e; z)|^s \geq \mathbb{E}_\omega |v_y - z|^s |G_\omega(x,y;z)|^s$$

$$\overset{(II)}{=} \mathbb{E}_\omega |v_y - z|^s \frac{|\alpha|^s}{|v_y - \beta|^s}$$

$$= \int_{\mathbb{Z}^d\setminus\{y\}} dP\ |\alpha|^s \int dv_y\, \varrho(v_y) \frac{|v_y - z|^s}{|v_y-\beta|^s}$$ lengthy but elementary

$$\geq \int_{\mathbb{Z}^d\setminus\{y\}} dP\ |\alpha|^s\, \underbrace{C' \frac{\|\varrho\|_1^s}{\|\varrho\|_\infty^s}} \int dv_y\, \varrho(v_y) \frac{1}{|v_y-\beta|^s}$$

$$\overset{(\mathrm{II})}{=} C \frac{\|\varrho\|_1^s}{\|\varrho\|_\infty^s}\ \mathbb{E}_\omega\, |G_\omega(x,y;z)|^s$$

$$\Rightarrow \mathbb{E}_\omega\, |G_\omega(x,y;z)|^s \leq (C^{-1}\|\varrho\|_\infty^s) \sum_{|e|=1} \mathbb{E}_\omega\, |G_\omega(x,y+e;z)|^s.$$

If $x \neq y+e$ this can be repeated. (cf. $y \neq x$ above!)

Iterating $|x-y|$ times: the distance of x and y

$$\mathbb{E}_\omega\, |G_\omega(x,y;z)|^s \leq (C^{-1}\|\varrho\|_\infty^s)^{|x-y|} \sum_{i=1}^{(2d)^{|x-y|}} \underbrace{\mathbb{E}_\omega\, |G_\omega(x,y_i;z)|^s}_{\substack{\leq C'\|\varrho\|_\infty^s \\ (\mathrm{Lm.}\ A)}}$$

$$\leq \underbrace{(C^{-1}\|\varrho\|_\infty^s)^{|x-y|} \cdot \underbrace{(2d)^{|x-y|}}_{\text{number of terms in the sum}}} \cdot C'\|\varrho\|_\infty^s$$

$$= C'\|\varrho\|_\infty^s\, e^{-m|x-y|}$$

with $e^{-m} = 2dC^{-1}\|\varrho\|_\infty^s$.

For $\|\varrho\|_\infty$ small enough, $m > 0$. ∎

<u>Proof of Lemma C:</u>

Have to show: $|\mathrm{Im}\, z|\ \mathbb{E}_\omega\, |G_\omega(x,y;z)|^2 \leq C e^{-m|x-y|}$,
(point to above) i.e. lift exponent from s to 2.

<u>Wiggle</u> the potential at $x \in \mathbb{Z}^d$:

$$h_{\omega,\kappa} := h_\omega + \kappa P_x = h_{\omega+\kappa\delta_x}.$$

On $\Omega \times \mathbb{R} \ni (\omega, \kappa)$ define

$$d\tilde{P}(\omega, \kappa) := \rho(v_x + \kappa)\, d\kappa\, dP(\omega).$$

By substitution and $\int \rho = 1$:

$$\int dP(\omega)\, f(\omega) = \int d\tilde{P}(\omega, \kappa)\, f(\omega + \kappa \delta_x).$$

<u>Claim:</u> (to be proven at the end)

$$|\operatorname{Im} z| \cdot |G_{\omega,\kappa}(x, y; z)|^2 \leftarrow \text{Green's fct. w.r.t. } h_{\omega,\kappa}$$

$$\leq \frac{|\operatorname{Im} G_\omega(x,x;z)^{-1}|}{|\kappa + G_\omega(x,x;z)^{-1}|^2} \cdot \frac{|G_\omega(x,y;z)|^s}{|G_\omega(x,x;z)|^s}.$$

Thus

$$\mathbb{E}_\omega |\operatorname{Im} z| \cdot |G_\omega(x,y;z)|^2$$

$$= \int dP(\omega)\, d\kappa\, \rho(v_x + \kappa)\, |\operatorname{Im} z| \cdot |G_\omega(x,y;z)|^2$$

by Claim

$$\leq \int dP(\omega)\, |G_\omega(x,y;z)|^s \underbrace{\int d\kappa\, \rho(v_x + \kappa) \frac{|\operatorname{Im} G_\omega(x,x;z)^{-1}|}{|\kappa + G_\omega(x,x;z)^{-1}|^2 \, |G_\omega(x,x;z)|^s}}_{\leq C, \text{ indep. of } \omega}$$

(esp. of v_x)

(elementary, lengthy)

(G_ω treated as arbitrary complex number, no special properties used)

$$\leq \mathbb{E}_\omega |G_\omega(x,y;z)|^s \cdot C$$

$$\overset{\text{Im.B}}{\leq} C e^{-m|x-y|}.$$

<u>Proof of Claim:</u>

(i) $$|\operatorname{Im} z| \cdot |G_{\omega,\kappa}(x,y;z)|^2 \leq |\operatorname{Im} z| \sum_{y' \in \mathbb{Z}^d} |G_{\omega,\kappa}(x,y';z)|^2$$

"$\sum_{y'} |\delta_{y'}\rangle\langle\delta_{y'}| = \mathbb{1}$"

$$= |\operatorname{Im} z| \langle \delta_x, (h_{\omega,\kappa} - z)^{-1} (h_{\omega,\kappa} - \bar{z})^{-1} \delta_x \rangle$$

$$= |\operatorname{Im} \langle \delta_x, (h_{\omega,\kappa} - z)^{-1} (h_{\omega,\kappa} - \bar{z})^{-1} (h_{\omega,\kappa} - \bar{z}) \delta_x \rangle|$$

$$= |\operatorname{Im} G_{\omega,\kappa}(x,x;z)|$$

by a resolvent identity:

$$= \frac{|\operatorname{Im} G_\omega(x,x;z)^{-1}|}{|\alpha + G_\omega(x,x;z)^{-1}|^2}.$$

(ii) $$|\operatorname{Im} z| \cdot |G_{\omega,\alpha}(x,y;z)|^2 \le \frac{|\operatorname{Im} G_\omega(x,x;z)^{-1}|}{|\alpha + G_\omega(x,x;z)^{-1}|^2} \cdot \frac{|G_\omega(x,y;z)|^2}{|G_\omega(x,x;z)|^2}.$$

(without proof here)

Combine using $\min(1,t^2) \le t^s \quad \forall t \ge 0$.

This concludes the proof of the claim, and thus of the Lemma. ∎

That's it, we have proven Anderson localization for large disorder on the lattice.