SPECTRAL ANALYSIS OF (SOME) SCHRÖDINGER OPERATORS II

1. Introduction

We continue our study of (some) Schrödinger operators $-\Delta + V$ on $L^2(\mathbb{R}^d)$. We aim to show two results:

(1) if V is in the Rollnik class, then the number of bound states is finite,
(2) under suitable assumptions on V, $-\Delta + V$ has no positive eigenvalues[1].

For each result we introduce an appropriate tool. For the first one, we will use the *Birman-Schwinger principle*, and for the latter we will use the *virial theorem.*

The Birman-Schwinger principle can also be used to derive Lieb-Thirring inequalities (see assignment). We will use it to prove also that in dimension 1 and 2, the Schrödinger operator $-\Delta + V$ has a bound state as long as $V \le 0$ is non-trivial. Sobolev inequalities ensure us that this result is false in higher dimensions.

We also emphasize that regarding the second problem, much more is known about the absence of embedded (in the essential spectrum) eigenvalues: we refer the reader for instance to [2][Thm XIII.58] and [1] where it can be found geenric conditions on V for the absence of positive eigenavlues. Note that there exist potentials that give rise to positive eigenvalues[2] In dimension $d \ge 5$, we exhibit an example of a Schrödinger operator with non-trivial kernel.

The virial method is a particular case of a more general technique that you can find in the litterature called Mourre-estimate technique.

2. Bound on the number of bound states

2.1. The Birman-Schwinger principle.

It relates eigenfunctions of $-\Delta + V$ with negative eigenvalues $E = -a^2 < 0$ with eigenfunctions of a compact operator, in the case when V is non-positive: $V \le 0$.

Lemma 1 (Birman-Schwinger principle)**.** *Let $V \le 0$ be a measurable potential such that the multiplication operator $\psi \mapsto |V|^{1/2}\psi$ is bounded.*

Then $E = -a^2 < 0$ is an eigenvalue of $-\Delta + V$ if and only if 1 is an eigenvalue of

$$K_a := |V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2}. \tag{1}$$

and if this is the case, they have the same multiplicity.

[1] that is in $(0, +\infty)$. Ruling out the possibility of a non-trivial kernel is another problem.

[2] they are called Neumann-Wigner potentials, as they were the first to exhibit an example of such potentials [4][pp. 291-293].

Remark 2. *We might be able to relax the constraint* $|V|^{1/2}$ *bounded, but it has to be checked each time.*

Furthermore, if V *is bounded, we can relax the condition* $V \leq 0$ *up to replacing* K_a *by*

$$|V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2}\mathrm{sign}(V).$$

Proof. Take an eigenfunction $\psi \in \mathrm{dom}(-\Delta + V) = H^2(\mathbb{R}^d)$: $(-\Delta + V)\psi = E\psi$. Rewriting the eigenequation, we have:

$$(-\Delta + a^2)\psi = -V\psi = |V|\psi.$$

Applying the resolvent $(-\Delta + a^2)^{-1}$ and afterwards $|V|^{1/2}$, we obtain:

$$|V|^{1/2}\psi = \big[|V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2}\big]|V|^{1/2}\psi,$$

hence 1 is an eigenvalue of K_a. Conversely, if ϕ satisfies $K_a\phi = \phi$, then setting $\psi := (-\Delta + a^2)^{-1}|V|^{1/2}\phi$, there holds:

$$(-\Delta + a^2 + V)\psi = |V|^{1/2}\phi - |V|^{1/2}\big[|V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2}\big]\phi = 0.$$

Hence $E = -a^2$ is an eigenvalue of $-\Delta + V$. The multiplicity are the same as:

$$\psi \mapsto (-\Delta + a^2)^{-1}|V|^{1/2}|V|^{1/2}\psi$$

is the identity on the eigenspace $\ker(-\Delta + V + a^2)$, and

$$\phi \mapsto |V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2}\phi$$

is the identity on $\ker(K_a - 1)$. □

2.2. The Birman-Schwinger bound.

Theorem 3. *On* $\mathbb{R}^3$*, let* $V \in \mathcal{R}$*, and let* $V = V_+ - V_-$ *be its splitting into positive and negative parts:*

$$V_\pm = \max(\pm V, 0).$$

Let $N(V)$ *be the number of negative eigenvalues of* $-\Delta + V$ *(counted with multiplicities). Then* $N(V)$ *is finite and there holds:*

$$N(V) \leq \frac{1}{(4\pi)^2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{V_-(x)V_-(y)}{|x-y|^2}\mathrm{d}x\mathrm{d}y.$$

Proof. First recall that V is $-\Delta$-compact, hence $\sigma_{\mathrm{ess}}(-\Delta + V) = [0, +\infty)$. *Restriction to* $V = -V_-$. The following quadratic form inequality holds: $-\Delta + V \geq -\Delta - V_-$ as $V \geq -V_-$. So by the min-max principle, we have for all $n \geq 1$

$$\mu_n(-\Delta - V_-) \leq \mu_n(-\Delta + V),$$

and so $N(V) \leq N(-V_-)$. More generally, writing $N_E(V)$ the number of eigenvalues (counted with multiplicities) of $-\Delta + V$ **strictly** smaller than $E < 0$, we have: $N_E(V) \leq N_E(-V_-)$. So w.l.o.g. we can assume that $V = -V_- \leq 0$.

Restriction to $V_- \in C_0^\infty(\mathbb{R}^3)$. Consider a compactly supported mollifier $(\phi_\varepsilon)_{\varepsilon>0}$, that is $\phi_\varepsilon(x) = \varepsilon^{-3}\phi_1(x/\varepsilon)$ where $\phi_1 \in C_0^\infty(\mathbb{R}^3, [0,1])$ with $\int \phi_1 = 1$.

Then $(V_- * \phi_\varepsilon)_{\varepsilon>0}$ is a family[3] in $C_0^\infty(\mathbb{R}^3)$ which converges to V_- as ε tends to 0 (in $\mathcal{R}$ and in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$).

We emphasize that $\mathcal{R} \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)$ as we have:

$$\begin{aligned}\iint \frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y &\geq \int_x |V(x)| \int_{y:|x-y|\leq 1} \frac{|V(y)|}{|x-y|^2}\mathrm{d}y\mathrm{d}x,\\ &\geq \int_x |V(x)| \int_{y:|x-y|\leq 1} |V(y)|\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By definition of the μ_n's, for all $n \geq 1$ we have:

$$\lim_{\varepsilon\to 0^+} \mu_n(-V_- * \phi_\varepsilon) = \mu_n(-V_-).$$

Hence for all $n \in \mathbb{N}$ and all $E < 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ we have:

$$N_E(-V_-) \leq N_E(-V_- * \phi_\varepsilon)$$

and

$$N(-V_-) \leq \liminf_{\varepsilon\to 0^+} N(-V_- * \phi_\varepsilon).$$

So w.l.o.g. we can assume that $V = -V_-$ is smooth and compactly supported.

Application of the Birman-Schwinger principle. For $\lambda \geq 0$, let $\mu_n(\lambda) := \mu_n(-\Delta + \lambda V)$.

As $V \leq 0$ is $-\Delta$-compact, the function $\lambda \geq 0 \mapsto \mu_n(\lambda)$ is monotone decreasing.

Looking at the definition of the μ_n's, we easily see that $\mu_n(\lambda)$ is also a continuous function in λ.

We have $\mu_n(0) = 0$. So if $\mu_n(1) < E$, then by continuity, there exists $0 < \lambda < 1$ such that $\mu_n(\lambda) = E$ (necessarily unique[4]).

By the Birman-Schwinger principle, we have $\mu_n(\lambda) = E$ if and only if 1 is an eigenvalue of $(\lambda|V|)^{1/2}(-\Delta - E)^{-1}(\lambda|V|)^{1/2}$, that is if and only if $1/\lambda$ is an eigenvalue of

$$K_a := |V|^{1/2}(-\Delta + a^2)|V|^{1/2} \geq 0,\ a^2 = -E > 0.$$

[3] see any book on distributions.

[4] why?

As $1/\lambda > 1$, we get:

$$\begin{aligned} N_E(V) &= \mathrm{Card}\big\{n,\ \mu_n(1) < E\big\}, \\ &= \mathrm{Card}\big\{n,\ \exists\, 0 < \lambda < 1,\ \mu_n(\lambda) = E\big\}, \\ &= \sum_{\mu > 1,\ \text{eig. of } K_a} 1, \\ &\le \sum_{\mu > 1,\ \text{eig. of } K_a} \mu^2, \\ &\le \sum_{\mu \text{ eig. of } K_a} \mu^2 = \iint |K_a(x,y)|^2 \mathrm{d}x\mathrm{d}y, \end{aligned}$$

where the sum over the eigenvalues of K_a takes into account their multiplicities. A computation yields:

$$\iint |K_a(x,y)|^2 \mathrm{d}x\mathrm{d}y = \frac{1}{(4\pi)^2} \iint |V(x)| \frac{e^{-2a|x-y|}}{|x-y|^2} |V(y)| \mathrm{d}x\mathrm{d}y \underset{a\to 0^+}{\longrightarrow} \frac{1}{(4\pi)^2} \|V\|_{\mathcal{R}}^2.$$

Thus we get:

$$N(V) = \lim_{E\to 0^-} N_E(V) \le \frac{1}{(4\pi)^2} \iint \frac{|V(x)||V(y)|}{|x-y|^2} \mathrm{d}x\mathrm{d}y.$$

□

2.3. **Existence of bound states in lower dimension.**

Lemma 4. *In dimension $d = 1$ or $d = 2$, let $V \in C_0^\infty(\mathbb{R}^d)$ such that $V \le 0$. Then for all $\lambda > 0$, $-\Delta + \lambda V$ ha a negative eigenvalues.*

Proof. By the Birman-Schwinger principle, it suffices to show that for any $\lambda > 0$, there exists $\varepsilon > 0$ such that $K_\varepsilon = |V|^{1/2}(-\Delta + \varepsilon^2)|V|^{1/2}$ has an eigenvalue $1/\lambda_0 \ge 1/\lambda$. Indeed, this implies that $-\varepsilon^2$ is an eigenvalue of $-\Delta + \lambda_0 V$, and we conclude by the min-max principle.

As K_ε is compact[5], it suffices to show that $\lim_{\varepsilon\to 0^+} \|K_\varepsilon\|_{\mathcal{L}} = +\infty$ (the norm is the largest eigenvalue of $K_\varepsilon \ge 0$).

Take $\eta \in C_0^\infty(\mathbb{R}^d, [0, +\infty))$ with $|V|^{1/2}\eta \ne 0$. Writing $\phi = |V|^{1/2}\eta$, we have:

$$\begin{aligned} \langle \eta, K_\varepsilon \eta \rangle &= \langle \phi, (-\Delta + \varepsilon^2)^{-1} \phi \rangle, \\ &= \int \frac{|\hat{\phi}(p)|^2}{|p|^2 + \varepsilon^2} \mathrm{d}p \underset{\varepsilon\to 0^+}{\longrightarrow} \int \frac{|\hat{\phi}(p)|^2}{|p|^2} \mathrm{d}p = +\infty. \end{aligned}$$

The convergence holds by monotone convergence, and the latter integral is infinite as $\phi \in L^1(\mathbb{R}^d)$, hence $\hat{\phi} \in C_0^0(\mathbb{R}^d)$ and $\hat{\phi}(0) = (2\pi)^{d/2} \int \phi > 0$. The argument uses the fact that $|p|^{-2}$ is not locally integrable around 0. □

Remark 5. *We emphasize that the result is flase in dimension $d \ge 3$. Indeed, by Sobolev inequality, there holds:*

$$\int |V||\psi|^2 \le \left(\int |V|^{d/2} \right)^{2/d} \left(\int |\psi|^{2d/(d-2)} \right)^{(d-2)/d} \le C \|V\|_{L^{d/2}(\mathbb{R}^d)} \|\nabla\psi\|_{L^2(\mathbb{R}^d)}^2.$$

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[5] it is of the form $f(x)g(-i\nabla)^2 f(x)$ where $f, g \in L^\infty(\mathbb{R}^d)$ both tend to 0 at infinity.

3. The virial theorem

3.1. **The S.C.G.U. of dilations.** Let $a > 0$. The dilation operator U_a is the unitary operator defined by

$$\forall \psi \in L^2(\mathbb{R}^d),\ (U_a\psi)(x) := a^{d/2}\psi(ax).$$

It is not difficult to see that $U_a^* = U_{a^{-1}}$. Via the logarithm $\log : (0, +\infty) \to \mathbb{R}$, we obtain a S.C.G.U. $(U_{e^t})_{t\in\mathbb{R}}$. That it is strongly continuous is easy to establish using the density (in $L^2(\mathbb{R}^d)$) of the set $C_c(\mathbb{R}^d)$ of continuous functions with compact support.

By Stone theorem, it is associated to a unique self-adjoint operator, which is given by:

$$D = \frac{id}{2} + i\sum_{j=1}^{d} x_j\partial_{x_j} = \frac{1}{2}\{x, i\nabla\},$$

where $e^{-iD\log(a)} = U_a$.

3.2. **Statement.** Let V be a (real)-potential: a computation yileds:

$$(U_aVU_a^*) =: V_a,\ (V_a\psi)(x) = V(ax)\psi(x).$$

Similarly we have:

$$(U_a\Delta U_a^*) = a^{-2}\Delta.$$

Furthermore, for V regular enough we have:

$$\frac{V_a(x) - V(x)}{a-1} \underset{\substack{a\to 1\\ a\neq 1}}{\longrightarrow} x\cdot\nabla V(x) =: W(x).$$

Furthermore for a potential V, and $H = -\Delta + V$, there *formally* holds $i[D, H] = 2(-\Delta) - W$. Hence, if ψ is an eigenfunction of H: $H\psi = E\psi$, we *formally* have:

$$\begin{aligned}2\langle\psi, -\Delta\psi\rangle - \langle\psi, W\psi\rangle &= \langle\psi, i[D, H]\psi\rangle,\\ &= 0.\end{aligned}$$

This result constitutes the virial theorem, which we have to justify rigorously.

Theorem 6 (The Virial theorem)**.** *Let V a real-valued function seen as a multiplication operator on $L^2(\mathbb{R}^d)$ such that:*

(1) *it is $-\Delta$-bounded with relative bound (strictly) smaller than* 1.
(2) *there exists a multiplication operator W on $L^2(\mathbb{R}^d)$ with* $\mathrm{dom}(W) \supset \mathrm{dom}(-\Delta)$ *such that for all $\psi \in \mathrm{dom}(-\Delta)$, there holds:*

$$\lim_{\substack{a\to 1\\ a\neq 1}} \frac{V_a\psi - V\psi}{a-1} = W\psi \in L^2(\mathbb{R}^d).$$

Then if $\psi \in \mathrm{dom}(-\Delta)$ is an eigenfunction of H with eigenvalue E then:

$$2\langle\psi, -\Delta\psi\rangle = \langle\psi, W\psi\rangle = 2\langle\psi, (E - V)\psi\rangle \geq 0. \tag{2}$$

Proof. We emphasize that the first equation is used to argue by the Kato-Rellich theorem that $-\Delta + V$ (and $-\Delta + V_a$ for a close enough to 1) is self-adjoint with domain $\mathrm{dom}(-\Delta)$.

Conjugating the eigen-equation with U_a, and writing $\psi_a = U_a\psi$ we obtain:

$$\begin{aligned}(-\Delta + a^2V_a)\psi_a &= Ea^2\psi_a,\\ (-\Delta + V) &= E\psi.\end{aligned}$$

Taking the inner product with ψ in the first line and with ψ_a in the second, we get:

$$\begin{aligned}\langle(-\Delta + a^2V_a)\psi_a, \psi\rangle &= Ea^2\langle\psi_a, \psi\rangle,\\ \langle\psi_a, (-\Delta + V)\rangle &= E\langle\psi_a, \psi\rangle.\end{aligned}$$

Hence we have:

$$\begin{aligned}E(a^2 - 1)\langle\psi_a, \psi\rangle &= \langle(-\Delta + a^2)\psi_a, \psi\rangle - \langle\psi_a, (-\Delta + V)\psi\rangle,\\ &= \langle(a^2V_a - V)\psi_a, \psi\rangle.\end{aligned}$$

Dividing by $a - 1$ we obtain

$$E(a + 1)\langle\psi_a, \psi\rangle = (a + 1)\langle\psi_a, V_a\psi\rangle + \frac{1}{a-1}\langle\psi_a, (V_a - V)\psi\rangle,$$

and taking the limit $a \to 1, a \neq 1$ gives[6]

$$2E\langle\psi, \psi\rangle = 2\langle\psi, V\psi\rangle + \langle\psi, W\psi\rangle.$$

As $(-\Delta + V)\psi = E\psi$, it ends the proof. $\square$

Remark 7. *Typically we can establish the second condition of Thm 6 as follows. Assume that the convergence:*

$$\frac{V_a(x) - V(x)}{a - 1} \to W(x)$$

holds almost everywhere, and that there exists $\tilde{W}$ with $\mathrm{dom}(\tilde{W}) \supset \mathrm{dom}(-\Delta)$ such that:

$$\Big|\frac{V_a(x) - V(x)}{a - 1}\Big| \leq \tilde{W}(x).$$

Then we get: $\mathrm{dom}(W) \supset \mathrm{dom}(\tilde{W}) \supset \mathrm{dom}(-\Delta)$, and the second condition follows by dominated convergence.

3.3. **Absence of positive eigenvalues.** We now show how we can use the virial theorem to ensure the absence of positive eigenvalues.

Theorem 8. *Let V be a real-valued function which is $-\Delta$-bounded with relative bound (strictly) smaller than 1. Then $H = -\Delta + V$ has no positive eigenvalues if any of the following three conditions hold.*

(1) *V satisfies the conditions of Thm 6 and V is repulsive in the sense that[7]:*

$$\forall\, x \in \mathbb{R}^3 \;\&\; a > 1,\; V(ax) \leq V(x).$$

(2) *In dimension $d \geq 3$, V is homogoneous of degree $-\alpha$ with $0 < \alpha < 2$:*

$$V(ax) = a^{-\alpha}V(x),\; x \neq 0.$$

[6] as an exercise, justify the limit: what kind of convergence do we have for each function?

[7] formally $W(x) \leq 0$. Recall that a potential V gives rise to a force field $-\nabla V$.

(3) *V satisfies the conditions of Thm 6 and for some $b > 0$, we have the quadratic form inequality:*
$$-\Delta - \frac{1}{2}(1+b)W - bV \geq 0.$$

Proof. If (ii) holds, let us show that the result of Thm 6 still holds. First, by Hardy's inequality, we know that we can apply the KLMN theorem to define $-\Delta + V$ and $-\Delta + V_a$ for a close to 1. Then observe that for all $x \neq 0$ there holds:
$$\frac{V(ax) - V(x)}{a-1} = \frac{a^{-\alpha} - 1}{a-1} V(x) \underset{\substack{a \to 1 \\ a \neq 1}}{\longrightarrow} -\alpha V(x).$$
It suffices to follow step by step the proof of Thm 6. In particular we have the convergence:
$$(a+1)\langle \psi_a, V_a \psi \rangle + \frac{1}{a-1}\langle \psi_a, (V_a - V)\psi \rangle \underset{\substack{a \to 1 \\ a \neq 1}}{\longrightarrow} 2\langle \psi, V\psi \rangle + \langle \psi, W\psi \rangle,$$
where we have used the fact that $\|\psi_a - \psi\|_{H^1} \to 0$ for $\psi \in H^1(\mathbb{R}^d)$.

In any of the three cases, by the Virial theorem, for an eigenfunction ψ of H, we have:
$$\langle \psi, -\Delta\psi \rangle = \frac{1}{2}\langle \psi, W\psi \rangle.$$
If (i) holds: then $W \leq 0$ as an operator and $\psi \in \ker(-\Delta) = \{0\}$, which is absurd. So $-\Delta + V$ has no eigenvalues.

If (ii) holds, then $W = -\alpha V$, and we have:
$$\begin{aligned} 2E\langle \psi, \psi \rangle &= 2\langle \psi, (-\Delta + V)\psi \rangle, \\ &= (2-\alpha)\langle \psi, V\psi \rangle, \\ &= -\alpha^{-1}(2-\alpha)\langle \psi, W\psi \rangle, \\ &= -2\alpha^{-1}(2-\alpha)\langle \psi, -\Delta\psi \rangle < 0. \end{aligned}$$
If (iii) holds, then from the eigen-equation we obtain:
$$\begin{aligned} -Eb\langle \psi, \psi \rangle &= -b\langle \psi, (-\Delta + V)\psi \rangle + 0, \\ &= -b\langle \psi, (-\Delta + V)\psi \rangle + (1+b)\langle \psi, (-\Delta - \tfrac{1}{2}W)\psi \rangle, \\ &= \langle \psi, (-\Delta - \tfrac{1}{2}(1+W) - bV)\psi \rangle \geq 0, \end{aligned}$$
hence $E \leq 0$. □

3.4. **Examples.**

Atomic Hamiltonian. The main example is the atomic Schrödinger operator given by:
$$H_N^Z := \sum_{j=1}^{N} -\Delta_{x_j} - \frac{Z}{|x_j|} + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|},$$
where $N \in \mathbb{N}$ is the number of particles and $Z > 0$ is the total charge of the core at 0. The operator H_N^Z acts on $H^2(\mathbb{R}^{3N})$ and above $x_j \in \mathbb{R}^3$.

It is of the form $-\Delta + V$ where V is homogoneous of degree -1, hence it has no positive eigenvalues.

Small smooth potentials. Let $V \in C_0^\infty(\mathbb{R}^d)$ with $d \geq 3$. Then we have that $W = -x \cdot \nabla V$ is also in $C_0^\infty(\mathbb{R}^d)$, hence for $\varepsilon > 0$ given there exists $\lambda_0 > 0$ so that for $-\lambda_0 \leq \lambda \leq \lambda_0$ we have:

$$-\Delta - \tfrac{1}{2}(1+\varepsilon)\lambda x \cdot \nabla V - \lambda\varepsilon V \geq 0.$$

In particular this implies that $-\Delta + \lambda V$ has no positive eigenvalues.

Existence of a non-trivial kernel in dimension $d \geq 5$. Let us show that in dimension $d \geq 5$ there exist $a, R > 0$ such that $-\Delta - a\chi_{B(0,R)}$ has a non-trivial kernel. This corresponds to a suqare well in high dimension.

The eigen-equation is:

$$\begin{cases} -\Delta\psi &= 0 \quad \text{on} \quad B(0,R)^c, \\ -\Delta\psi &= a\psi \quad \text{on} \quad B(0,R). \end{cases}$$

If we look at radial solution, the equation on $B(0,R)^c$ gives[8] $\psi(x) = C|x|^{d-2}$ for $|x| \geq R$. Up to normalizing we can assume $C = 1$.

The equation on $B(0,R)$ gives $-\Delta\psi = \varepsilon\psi$, and to obtain an element on $H^2(\mathbb{R}^d)$ we must have $\psi_{|S(0,R)}$ and $\frac{\partial}{\partial n}\psi_{|S(0,R)}$ on both sides of the hypersphere $S(0,R)$ (where n is the outward normal: $n(x) = \frac{x}{|x|}$ for $x \in S(0,R)$).

We look solutions on $B(0,R)$ which are eigenfunctions of the Robin Laplacian:

$$\begin{cases} -\Delta\psi &= a\psi \quad \text{on} \quad B(0,R), \\ \frac{\partial}{\partial n}\psi &= -\alpha\psi \quad \text{on} \quad S(0,R), \end{cases}$$

with $\alpha > 0$ to be chosen. It is the unique self-adjoint operator corresponding to the closed non-negative quadratic form on $H^1(B(0,R))$:

$$\int_{B(0,R)} |\nabla\psi|^2 + \alpha \int_{S(0,R)} |\psi_{|S(0,R)}(y)|^2 \mathrm{d}_{S(0,R)}(y).$$

The eigenfunction[9] $\phi_1(|x|) \in H^2(B(0,R))$ corresponding to the lowest eigenvalue λ_1 is radial, and can be chosen positive. We choose $a = \lambda_1$. The condition across $S(0,R)$ gives:

$$\begin{aligned} -\alpha R^{-(d-2)} &= -\alpha(\phi_1(x))_{|x|\to R^-}, \\ &= (\tfrac{\partial}{\partial n}\phi_1(x))_{|x|\to R^-}, \\ &= (\tfrac{\partial}{\partial n}|\cdot|^{-(d-2)})_{|x|\to R^+} = -(d-2)R^{-(d-1)}. \end{aligned}$$

We thus choose $\alpha := \frac{d-2}{R} > 0$, and we thus obtain a non-trivial element of $\ker(-\Delta - \lambda_1\chi_{B(0,R)})$.

References

[1] Herbert Koch and Daniel Tataru, *Carleman estimates and absence of embedded eigenvalues*, Comm. Math. Phys. **267** (2006), no. 2, 419–449, DOI 10.1007/s00220-006-0060-y. MR2252331

[2] Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York-London, 1972.

[8] It suffices to write down the equation in spherical coordinates. The function is supposed to be radial, hence we obtain a Dirichlet problem on $B(0,R)^c$ by assigning the value of ψ on $S(0,R)$. The condition on the dimension is to ensure the square integrability.

[9] We claim this regularity $H^2(B(0,R))$, it can be proven, but we do not in this course.

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