Solution Ass. 2, Problems 3–5

①

3a

$$\|f\|_{L^q} \le C_{n,p,q} \|\nabla f\|_{L^p}.$$

$f_\lambda(x) = f(\lambda x)$

$$\|f_\lambda\|_{L^q} = \left(\int d^n x \, |f(\lambda x)|^q \right)^{1/q}$$

$y = \lambda x$

$d^n y = \lambda^n d^n x$

$\Rightarrow d^n x = \lambda^{-n} d^n y$

$$= \left(\int d^n y \, \lambda^{-n} |f(y)|^q \right)^{1/q} = \lambda^{-n/q} \|f\|_{L^q}$$

$$\|\nabla f_\lambda\|_{L^p} = \left(\int d^n x \, |\nabla f_\lambda(x)|^p \right)^{1/p}$$

$$= \left(\int d^n x \, \lambda^p |\nabla f(\lambda x)|^p \right)^{1/p}$$

$$= \lambda \, \lambda^{-n/p} \|\nabla f\|_{L^p}.$$

$$\Rightarrow \quad \lambda^{-n/q} \overset{!}{=} \lambda^{1-n/p}$$

$$\Rightarrow \quad -\frac{n}{q} = 1 - \frac{n}{p} \quad \Longrightarrow \quad \boxed{\frac{n}{p} = 1 + \frac{n}{q}}$$

E.g. $n=3$, $p=2$:

$\frac{3}{2} = 1 + \frac{3}{q}$

$\frac{1}{2} = \frac{3}{q} \Rightarrow \underline{q=6.}$

3b $u \in H^m(\mathbb{R}^n)$, $m > \frac{n}{2}$.

<u>Claim:</u> $\|u\|_{L^\infty} \leq C_{m,n} \|u\|_{H^m}$.

$$u(x) = \int dp \frac{1}{(2\pi)^{n/2}} e^{ip\cdot x} \hat{u}(p)$$

$$= \int dp \frac{1}{(2\pi)^{n/2}} e^{ip\cdot x} \hat{u}(p) (1+p^2)^{m/2} \frac{1}{(1+p^2)^{m/2}}$$

$$\leq \frac{1}{(2\pi)^{n/2}} \left\{ \int dp \, |e^{ip\cdot x}|^2 \, |\hat{u}(p)|^2 (1+p^2)^m \right\}^{1/2} \left\{ \int d^n p \frac{1}{(1+p^2)^m} \right\}^{1/2}$$

(3)

$$= C_n \|u\|_{H^m} \left\{ \int d^n p \, \frac{1}{(1+p^2)^m} \right\}^{1/2}$$

only one question: does this have sufficient decay at $p \to \pm\infty$?

Answer: yes, because

$$C_n \int_0^\infty d|p| \; \underbrace{|p|^{n-1} \frac{1}{(1+|p|^2)^m}}_{\sim\, p^{n-1-2m}}$$

We need: $\cancel{1} + 2m - n > \cancel{1}$

$$2m > n$$

$$\boxed{m > \frac{n}{2}.}$$

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4a $H_0 = -\frac{\Delta}{2}$ in $L^2(\mathbb{R}^3)$, $E \in \mathbb{R}$.

show: $\underset{\varepsilon \downarrow 0}{\text{s-lim}}\; \varepsilon (H_0 - E + i\varepsilon)^{-1} = 0$.

$$\| \mathcal{F}\left[\varepsilon (H_0 - E + i\varepsilon)^{-1} \varphi \right](p) \|_{L^2}^2$$

$$= \| \varepsilon \left(\frac{p^2}{2} - E + i\varepsilon \right)^{-1} \hat{\varphi}(p) \|_{L^2}^2$$

$$= \int \left| \frac{\varepsilon}{\frac{p^2}{2} - E + i\varepsilon} \right|^2 |\hat{\varphi}(p)|^2 \, dp$$

$$= \int \frac{\varepsilon^2}{\left[\left(\frac{p^2}{2} - E \right) + i\varepsilon \right] \left[\left(\frac{p^2}{2} - E \right) - i\varepsilon \right]} |\hat{\varphi}(p)|^2 \, dp$$

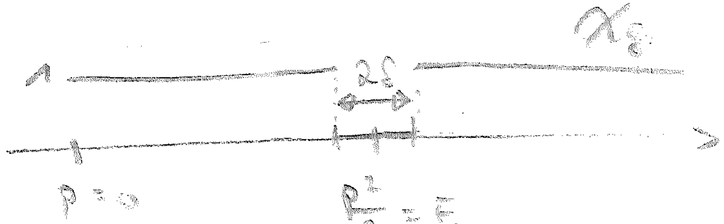
$$= \int \frac{\varepsilon^2}{\left| \frac{p^2}{2} - E \right|^2 + \varepsilon^2} |\hat{\varphi}(p)|^2 \, dp .$$

It is enough to check on a dense set because:

$$\frac{\varepsilon^2}{\left|\frac{p^2}{2}-E\right|^2+\varepsilon^2} \leq \frac{\varepsilon^2}{\varepsilon^2} = 1.$$

(i.e. $\| \varepsilon (H_0 - E + i\varepsilon)^{-1} \|_{op} \leq 1$ (uniformly in ε!)).

Take following dense set:

$X = \{ \varphi_\delta :$  $, \varphi \in L^2,$

$\hat{\varphi}_\delta = \chi_\delta \hat{\varphi} \}.$

By dominated convergence: $\varphi_\delta \longrightarrow \varphi \ (\delta \to 0).$

On supp φ_δ: $\left|\frac{p^2}{2} - E\right| > \delta^2.$

(6)

So: $\| \varepsilon (H_0 - E + i\varepsilon)^{-1} \varphi \|_{L^2}^2$

$$\le \int \frac{\varepsilon^2}{\delta^2 + \varepsilon^2} |\hat{\varphi}(p)|^2 \, dp$$

$$\le \frac{1}{\delta^2} \int \varepsilon^2 |\hat{\varphi}(p)|^2 \, dp \le \frac{1}{\delta^2} \varepsilon^2 \|\varphi\|_{L^2}^2 \longrightarrow 0 \quad (\varepsilon \to 0).$$

$\blacksquare$

4b) $\varphi : [0, \infty) \longrightarrow X$ continuous.

X Banach space.

$\varphi_\infty = \lim_{t \to \infty} \varphi(t)$ exists:

??: $\varphi_\infty = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} \varphi(t) \, dt.$

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$$\left\| e_\infty - \varepsilon \int_0^\infty e^{-\varepsilon t}\, e(t)\, dt \right\|$$

$$= \left\| \varepsilon \int_0^\infty e^{-\varepsilon t} \big(e_\infty - e(t)\big)\, dt \right\|$$

$$\leq \varepsilon \int_0^\infty e^{-\varepsilon t} \| e_\infty - e(t) \|\, dt$$

Note

$$\varepsilon \int_0^\infty e^{-\varepsilon t}\, dt = \varepsilon \frac{1}{-\varepsilon}(0 - e^0) = 1$$

Let $\delta > 0$. We know: $\exists T > 0$ s.t. $\| e_\infty - e(t) \| < \delta/2$ $\forall t > T$.

Consequently:

$$\leq \varepsilon \int_0^T e^{-\varepsilon t} \| e_\infty - e(t) \|\, dt + \underbrace{\varepsilon \int_T^\infty e^{-\varepsilon t}\, \delta\, dt}_{= \delta/2}$$

$$\leq \underbrace{\varepsilon \int_0^T e^{-\varepsilon t}\, dt}_{\leq T \left(\text{because } e^{-\varepsilon t} \leq 1\right)} \underbrace{\left\{ \| e_\infty \| + \sup_{t \in [0,T]} \| e(t) \| \right\}}_{\leq C_T \text{ by continuity}}$$

$\Rightarrow \; \leq \varepsilon \underbrace{T C_T}_{\text{fixed}} + \frac{\delta}{2} < \delta$ for ε small enough.

$\blacksquare$

4c | $H = H^*$, $D(H) = D(H_0)$, $H_0 = -\frac{\Delta}{2}$.)

$\Omega_+ = s\text{-}\lim_{t \to \infty} e^{iHt} e^{-iH_0 t}$ exists.

$\mathcal{H} = \mathcal{H}_B \oplus \operatorname{ran} \Omega_+$.

Show: $\forall \varphi \in \mathcal{H}: \quad \Omega_+^* \varphi = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iHt} \varphi \, dt$.

Write $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in \mathcal{H}_B$, $\varphi_2 \in \operatorname{ran} \Omega_+$.

By linearity we can consider both parts separately.

1) For $\varphi \in \mathcal{H}_B$: from lecture we know that

$\Omega_+^* = 0$ on $\mathcal{H}_B = (\operatorname{ran} \Omega_+)^\perp$.

So we have to show that R.H.S. $\to 0$.

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iHt} \varphi.$$

First we assume: $H\varphi = E\varphi$.

Then:

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iEt} \varphi$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon \, (H_0 - E + i\varepsilon)^{-1} \varphi = 0.$$

If $\varphi \in \mathcal{H}_B$, but only approximated by eigenvectors:

Due to unitarity & $e^{-\varepsilon t}$,

the expression $\| \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iHt} \varphi \, dt \|$

$$\leq \varepsilon \int_0^\infty e^{-\varepsilon t} dt \, \|\varphi\| \leq 1 \|\varphi\|,$$

i.e. a bounded operator.

So it commutes with limits:

$$\epsilon \int_0^\infty e^{-\epsilon t} e^{iH_0 t} e^{-iHt} \left(\lim_{n\to\infty} \varphi_n\right) dt$$

$$= \lim_{n\to\infty} \underbrace{\epsilon \int_0^\infty e^{-\epsilon t} e^{iH_0 t} e^{-iHt} \varphi_n \, dt}_{=0} = 0.$$

2) Remains the case $\varphi \in \operatorname{Ran} \Omega_+$. Have to show that $(\text{RHS})\varphi = \psi$

By (b) it is enough to show that

$\varphi = \Omega_+ \psi$,

$$\lim_{t\to\infty} e^{iH_0 t} e^{-iHt} \varphi \text{ exists.}$$

Write $\varphi = \Omega_+ \psi$, $\psi \in \mathcal{H}$.

Then we can use the following:

$$e^{iH_0 t} e^{-iHt} \Omega_+ \varphi$$

$$= e^{iH_0 t} \Omega_+ e^{-iH_0 t} \varphi$$

$$= \Omega_+ \underbrace{\left(e^{iHt} e^{-iH_0 t} \varphi \right)}_{\text{exists, since the wave operators exist.}}$$

$\Rightarrow$ the integral on the r.h.s. exists and is equal to $\lim_{t \to \infty} e^{iH_0 t} e^{-iHt} \varphi$.

Claim: this is equal to Ω_+^*.

Well:

$$\langle \psi, \lim_{t \to \infty} e^{iH_0 t} e^{-iHt} \varphi \rangle$$

$$= \lim_{t \to \infty} \langle e^{iHt} e^{-iH_0 t} \psi, \varphi \rangle = \langle \Omega_+ \psi, \varphi \rangle$$

$\forall \psi \in \mathcal{H}$. $\blacksquare$

5a Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ an s.a. op. in $\mathcal{H}$.

$B_n := in\,(A + in)^{-1}$, $n \in \mathbb{Z}$.

$A_n := B_n A B_{-n}$

$U_n(t) := e^{-iA_n t}$.

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show: by u.l. it is enough to show: $\lim_{n\to\infty} U_n(t)\varphi$ exists $\forall \varphi \in D(A)$.

Proof: $U_n(t)\varphi - U_m(t)\varphi = \int_0^t \frac{d}{ds}\{U_n(s)U_m(t-s)\varphi\}\,ds$

$= -i\int_0^t U_n(s)\,(A_n - A_m)\,U_m(t-s)\,\varphi\,ds$

we use for this exchange: B_n consists of resolvents of A, modulo domains A and $(A+i\mu)^{-1}$ can be exchanged

→ write series for U_n and check explicitly.

$= -i\int_0^t U_n(s)\,U_m(t-s)\,(A_n - A_m)\,\varphi\,ds$

$\Rightarrow \|U_n(t)\varphi - U_m(t)\varphi\| \le |t|\,\|A_n\varphi - A_m\varphi\|$.

Since $A_n e \longrightarrow Ae$, by Cauchy criterion, we conclude that the limit $U_n(t)e$, $n \to \infty$, exists.

5b) Show that the generator of $U(t)$ is the operator A.

Let $e \in D(A)$. Then

$$\frac{U(t)e - e}{t} = \lim_{n\to\infty} \frac{1}{t}\left(U_n(t)e - e\right)$$

$$= \lim_{n\to\infty} \frac{1}{t}\int_0^t U_n(s) A_n e \, ds$$

$$\overset{(i)}{=} \frac{1}{t}\int_0^t U(s) Ae \, ds \xrightarrow{(ii)} U(0)Ae = Ae.$$

Why (ii)? $\left\| \frac{1}{t}\int_0^t U(s) Ae \, ds - Ae \right\|$

$$\leq \frac{1}{t}\int_0^t \left\| [U(s) - U(0)] Ae \right\| ds$$

Let $\varepsilon > 0$. $\exists \delta > 0 : t \in (0,\delta) \Rightarrow \|U(s) A e - U(0) A e\| < \varepsilon$.

$$\Rightarrow \frac{1}{t} \int_0^t \| [U(s) - U(0)] A e \| \, ds < \frac{1}{t} \int_0^t \varepsilon \, ds = \frac{t}{t} \varepsilon = \varepsilon.$$

Why (:)?

$$\left\| \frac{1}{t} \int_0^t U_n(s) A_n e \, ds - \frac{1}{t} \int_0^t U(s) A e \, ds \right\|$$

$$\le \frac{1}{t} \int_0^t \| U_n(s) A_n e - U(s) A e \| \, ds \longrightarrow 0$$

because $\| U_n(s) A_n e - U(s) A e \| \le \| [U_n(s) - U(s)] A e \| + \| U_n(s) [A_n e - A e] \|$,

both parts $\to 0$ by strong convergence.

This hold for $e \in D(A)$. $\Longrightarrow$ $A \subset B$, where B is the generator.

$A = A^{**}$, $B = B^{**}$,

$A \subset B$ and $\underset{\displaystyle B}{\overset{\displaystyle B^{*}}{\shortparallel}} \subset \underset{\displaystyle A}{\overset{\displaystyle A^{*}}{\shortparallel}}$ $\Rightarrow$ $A = B$.

$\blacksquare$