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Advanced Mathematical Physics

Assignment 2 of 4

To be handed in on Friday, May 19, at the beginning of the seminar.

Problem 1: Orthogonal Complement (3+3+3 points)

Let $\mathcal{H}$ be a Hilbert space and $M \subset \mathcal{H}$ a subset. Prove the following facts:

a. The orthogonal complement $M^\perp$ is a closed subspace.

b. $M \subset (M^\perp)^\perp$. If M is a subspace: $(M^\perp)^\perp = \overline{M}$.

Hint: You may use the fact that given a closed subspace $X \subset \mathcal{H}$, for every $x \in \mathcal{H}$ there exists a unique decomposition $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in X^\perp$.

c. $\left(\overline{M}\right)^\perp = M^\perp$.

Problem 2: The Coulomb potential (5+5 points)

a. Suppose that $V \in L^2 + L^\infty(\mathbb{R}^3)$.
Show that the L^2-part can be made arbitrarily small, i. e., for every $\varepsilon > 0$ there exist $V_2^\varepsilon \in L^2(\mathbb{R}^3)$ and $V_\infty^\varepsilon \in L^\infty(\mathbb{R}^3)$ such that $V = V_2^\varepsilon + V_\infty^\varepsilon$ and $\|V_2^\varepsilon\|_2 < \varepsilon$.

b. Show that the Coulomb potential, defined by $V(x) = \frac{1}{|x|}$ for $x \neq 0$ (and $V(0) = 0$), is in $L^2 + L^\infty(\mathbb{R}^3)$.

Problem 3: Sobolev inequalities (5+5 points)

a. Assume that for functions defined on $\mathbb{R}^n$ a Sobolev inequality of the following form holds: There exists a constant $C_{n,p,q}$ such that for all f we have

$$\|f\|_{L^q} \leq C_{n,p,q} \|\nabla f\|_{L^p}.$$

Given n and p, consider a rescaling $f_\lambda(x) = f(\lambda x)$ by a parameter $\lambda > 0$ to determine the only possible exponent q for which this can hold.

Remark: We write

$$\|\nabla f\|_{L^p}^p = \int |\nabla f(x)|^p \mathrm{d}x,$$

where $|\cdot|$ stands for any norm on the space $\mathbb{C}^n$; since all norms on the finite dimensional space $\mathbb{C}^n$ are equivalent[1], different choices yield different but equivalent L^p-norms—you are free to make a convenient choice.

b. Let $u \in H^m(\mathbb{R}^n)$ with $m > n/2$. Use the Fourier transform to show that $u \in L^\infty(\mathbb{R}^n)$, and that the following Sobolev inequality holds:

$$\|u\|_{L^\infty} \leq C_{m,n}\|u\|_{H^m}, \tag{1}$$

where the constant $C_{m,n}$ depends only on m and n.

Problem 4: Abelian Limits and Wave Operator (3+3+4 points)

a. Let $H_0 = -\Delta/2$ in $L^2(\mathbb{R}^3)$ and $E \in \mathbb{R}$. Show that

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} \varepsilon(H_0 - E + i\varepsilon)^{-1} = 0.$$

Hint: Use the Fourier transform and construct a convenient dense subspace for taking the limit.

b. Let $\varphi : [0,\infty) \to X$ be continuous, X a Banach space and assume that $\varphi_\infty := \lim_{t\to\infty} \varphi(t)$ exists. Prove that

$$\varphi_\infty = \lim_{\varepsilon\downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t}\varphi(t)\mathrm{d}t.$$

c. Let H be a self-adjoint operator in $L^2(\mathbb{R}^3)$ with $D(H) = D(H_0)$, $H_0 = -\Delta/2$, and assume that the wave operator $\Omega_+ = \operatorname{s-lim}_{t\to\infty} e^{iHt}e^{-iH_0t}$ exists. Assume furthermore that asymptotic completeness holds, i.e. $\operatorname{ran}\Omega_+ = \mathcal{H}_B^\perp$ (where $\mathcal{H}_B$ is the closure of the span of the eigenstates of H).

Prove that for all $\varphi \in \mathcal{H}$ we have

$$\Omega_+^*\varphi = \lim_{\varepsilon\downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t}e^{iH_0t}e^{-iHt}\varphi\,\mathrm{d}t.$$

Problem 5: On the Existence Proof for SCUGs (5+5 points)

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

[1] If $|\cdot|_A$ and $|\cdot|_B$ are two norms on a finite dimensional space X, then there exist constants $c, C > 0$ such that $c|x|_A \leq |x|_B \leq C|x|_A$ for all $x \in X$. In particular the generated topologies are the same, and Cauchy sequences of $|\cdot|_A$ are also Cauchy sequences of $|\cdot|_B$ and vice versa.

In Theorem 4.4 of the lecture notes we defined $B_m := im(A + im)^{-1}$ ($m \in \mathbb{Z}$) and $A_m := B_m A B_{-m}$. We also defined

$$U_m(t) := e^{-iA_m t} := \sum_{k \in \mathbb{N}} \frac{1}{k!} \left(-itA_m\right)^k .$$

We claimed that $U(t) = \text{s-lim}_{m\to\infty} U_m(t)$ is a strongly continuous unitary group generated by A. In this problem we are going to verify this claim.

a. You can take for granted that $U_m(t)$ is a SCUG for every $m \in \mathbb{N}$.

Show that the limit of $U_m(t)\varphi$ ($m \to \infty$) exists for all $\varphi \in D(A)$ and all $t \in \mathbb{R}$. Why does $U(t) := \text{s-}\lim_{m\to\infty} U_m(t)$ exist?

b. You can take for granted that $U(t)$ is a SCUG.

Show that its generator is the operator A.