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Advanced Mathematical Physics

Assignment 1 of 4

To be handed in on Friday, May 5, at the beginning of the seminar.

Problem 1: Self-adjoint extensions (5+5 points)

a. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If $A \subset B$, then $B^* \subset A^*$.

b. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If A is symmetric and B is a self-adjoint extension of A, then $A \subset B \subset A^*$

Problem 2: Graph of the adjoint operator (5+5 points)

Let $\mathcal{H}$ be a Hilbert space and A a self-adjoint operator in $\mathcal{H}$. Consider the direct sum $\mathcal{H} \oplus \mathcal{H}$ equipped with the scalar product $\langle (x', y'), (x, y) \rangle_{\mathcal{H}\oplus\mathcal{H}} := \langle x', x \rangle_{\mathcal{H}} + \langle y', y \rangle_{\mathcal{H}}$.

a. Verify that $\mathcal{H} \oplus \mathcal{H}$ is indeed a Hilbert space (i. e. show that $\mathcal{H} \oplus \mathcal{H}$ satisfies the vector space axioms, show that $\langle \cdot, \cdot \rangle_{\mathcal{H}\oplus\mathcal{H}}$ satisfies the axioms of a scalar product, and show that $\mathcal{H} \oplus \mathcal{H}$ with the natural norm is complete).

b. Let $\Gamma_A \subset \mathcal{H} \oplus \mathcal{H}$ the graph of A. Show that

$$(\Gamma_A)^\perp = \{(x', y') \in \mathcal{H} \oplus \mathcal{H} : y' \in \mathcal{D}(A) \text{ and } x' = -Ay'\}\,.$$

Problem 3: Operator with trivial adjoint (2+6+2 points)

Let $\mathcal{H} = L^2(\mathbb{R})$ and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis[1] of $\mathcal{H}$. Define an operator $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ by $\mathcal{D} := C_0^\infty(\mathbb{R})$ and

$$Af := \sum_{n=0}^{\infty} f(n) e_n. \tag{1}$$

a. Show that: The series in (1) converges.

b. Show that: For any $g \in \mathcal{H}$, $g \neq 0$, the mapping $f \mapsto \langle g, Af \rangle$ is not continuous, as a function from $(\mathcal{D}, \|\cdot\|_{\mathcal{H}})$ to $\mathbb{C}$.

[1] A countable family $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called orthonormal basis if $\langle e_n, e_k \rangle = \delta_{n,k}$ and for all $x \in \mathcal{H}$ we have $x = \sum_{n=0}^{\infty} e_n \langle e_n, x \rangle$ (as a convergent series).
Example: For $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, in $\mathcal{H} = L^2((0, 2\pi))$, this becomes the ordinary Fourier series.

c. Show that the domain $D(A^*)$ is trivial, i. e. $D(A^*) = \{0\}$.

Problem 4: Resolvent of the free Laplacian (10 points)

Let $H_0 = -\Delta$ the Laplacian in $L^2(\mathbb{R}^3)$. Show that for $\varphi \in L^2(\mathbb{R}^3)$ and $\kappa > 0$ we have

$$\left(\left(H_0 + \kappa^2\right)^{-1} \varphi\right)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} \varphi(y) \mathrm{d}y.$$

Hints: First, recall (giving at least a short sketch) why a multiplication operator in Fourier space turns into a convolution with the inversely transformed function $\check{f}$:

$$\left(\mathcal{F}^{-1} T_f \mathcal{F} \varphi\right)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \check{f}(x-y) \varphi(y) \mathrm{d}y.$$

Then use spherical coordinates and the residue theorem with a convenient complex integration path to calculate $\check{f}$.

Remark 1: The function $\frac{e^{-\kappa|x|}}{4\pi|x|}$ is called Yukawa potential or free Green's function.

Remark 2: In electrodynamics or partial differential equations, one considers linear differential operators L and tries to find a so called Green's function G such that $(LG)(x) = \delta(x)$ (Dirac delta distribution). So from the meta point of view the Green's function G is the representation of the resolvent (inverse of L) as the kernel of an integral operator.

Problem 5: Proof of the Weyl criterion (10 points)

Let X be a Banach space, $A : D \subset X \to X$ an operator and $\lambda \in \mathbb{C}$.

Prove that: If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in D with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, and $\|(A-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\lambda \in \sigma(A)$.