DEF: Let $U : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ a SCUG.
The <u>generator</u> of $U$ is the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ def. by

$$D(A) = \left\{ \varphi \in \mathcal{H} : \lim_{t \to 0} \frac{U(t)\varphi - \varphi}{t} \text{ exists} \right\},$$ i.e. where the 'strong' derivative exists

$$A\varphi := i \lim_{t \to 0} \frac{U(t)\varphi - \varphi}{t} = i \frac{d}{dt} U(t)\varphi \Big|_{t=0}.$$

THM 4.2: (STONE 1932)

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be the generator of a SCUG $U : \mathbb{R} \to \mathcal{L}(\mathcal{H})$.
Then:

(i) $A$ is densely defined and $A = A^*$

(ii) $U(t) D(A) \subset D(A)$, and $\forall \varphi \in D(A), t \in \mathbb{R}$: $i \frac{d}{dt} U(t)\varphi = AU(t)\varphi = U(t)A\varphi$.

(iii) $U(t)$ is uniquely determined by having generator $A$.

(iv) $\forall z \in \mathbb{C}_\pm$, $\varphi \in \mathcal{H}$:

$$(A - z)^{-1}\varphi = i \int_0^{\pm\infty} e^{izt} U(t)\varphi \, dt.$$

RMK: The integral is a vector-valued Riemann integral (limit of Riemann sums), satisfies the 'normal' rules,

e.g. $\| \int \varphi(t)\, dt \| \leq \int \| \varphi(t) \| \, dt$ for $\varphi : \mathbb{R} \to \mathcal{H}$ continuous

e.g. $B \int \varphi(t)\, dt = \int B \varphi(t)\, dt$ for $B$ a <u>bounded</u> operator.

PROOF:

(ii) We show $U(t)D(A) \subset D(A)$. Let $\varphi \in D(A)$

$$\frac{i}{\varepsilon}(U(\varepsilon) - \mathbb{1})(U(t)\varphi)$$
$$= \frac{i}{\varepsilon}(U(t+\varepsilon) - U(t))\varphi$$
$$= U(t) \frac{i}{\varepsilon}(U(\varepsilon) - \mathbb{1})\varphi$$ ← $U(t)$ is bounded $\Rightarrow$ commutes with limit.
$$\xrightarrow[(\varepsilon \to 0)]{} U(t)A\varphi$$

$$\Rightarrow U(t)\varphi \in D(A).$$

Diff. eqn. easy to deduce.

(i) <u>We show $\overline{D(A)} = \mathcal{H}$:</u> Let $\varphi \in \mathcal{H}$. Let $\delta > 0$.

Let $\varphi_\delta := \frac{1}{\delta}\int_0^\delta U(s)\varphi \, ds$. Claim: $\varphi_\delta \xrightarrow[(\delta \to 0)]{} \varphi$.

In fact:

Let $\varepsilon > 0$ and $\delta$ s.th. $\|U(s)\varphi - \varphi\| < \varepsilon \quad \forall s \in [0, \delta)$.

Then $\|\frac{1}{\delta}\int_0^\delta U(s)\varphi \, ds - \varphi\| = \|\frac{1}{\delta}\int_0^\delta U(s)\varphi \, ds - \frac{1}{\delta}\int_0^\delta \varphi \, ds\|$

$\leq \frac{1}{\delta}\int_0^\delta \|U(s)\varphi - \varphi\| \, ds < \frac{1}{\delta}\int_0^\delta \varepsilon \, ds = \varepsilon$.

Remains to show: $\varphi_\delta \in D(A)$. $\quad U(t)\varphi_\delta = \frac{1}{\delta}\int_0^\delta U(t+s)\varphi \, ds = \frac{1}{\delta}\int_t^{t+\delta} U(r)\varphi \, dr$.

As the integral of a continuous function, this is differentiable w.r.t. $t$ at $t=0$.

By def., $D(A)$ is the set of vectors where $U(t)$ is differentiable. $\Rightarrow \varphi_\delta \in D(A)$.

<u>We show $A \subset A^*$:</u> Let $\varphi, \psi \in D(A)$. Recall: $\langle \varphi, \psi \rangle = \langle U(t)\varphi, U(t)\psi \rangle$.

Thus $0 = i\frac{d}{dt}\langle U(t)\varphi, U(t)\psi \rangle \Big|_{t=0} = \langle -A\varphi, \psi \rangle + \langle \varphi, A\psi \rangle$.

<u>We show $A^* \subset A$:</u> Let $\varphi \in D(A^*)$. Then for all $\eta \in D(A)$:

$\langle \eta, \frac{i}{t}(U(t) - \mathbb{1})\varphi \rangle = \langle \frac{i}{-t}(U(-t) - \mathbb{1})\eta, \varphi \rangle$

difference = integral over derivative

$= \frac{1}{-t}\int_0^{-t} \frac{d}{ds}\langle i U(s)\eta, \varphi \rangle \, ds$

$\overset{(ii)}{=} \frac{1}{-t}\int_0^{-t} \langle A U(s)\eta, \varphi \rangle \, ds$ ← $\eta \in D(A)$ by assumption

$= \frac{1}{-t}\int_0^{-t} \langle \eta, U(-s) A^*\varphi \rangle \, ds$ ← $\varphi \in D(A^*)$ by assumption

$= \langle \eta, \frac{1}{t}\int_0^t U(s) A^*\varphi \, ds \rangle$

since $\varphi$'s are dense $\Rightarrow \; i\,\frac{U(t)\varphi-\varphi}{t} = \frac{1}{t}\int_0^t U(s)A^*\varphi\,ds$

As before: $\frac{1}{t}\int_0^t U(s)A^*\varphi\,ds \xrightarrow[(t\to 0)]{} U(0)A^*\varphi = A\varphi$.

But by definition:

if $\lim_{t\to 0} i\,\frac{U(t)\varphi-\varphi}{t}$ exists, then it is $= A\varphi$ and $\varphi\in D(A)$.

So we have shown: $\varphi\in D(A^*) \Rightarrow \varphi\in D(A)$.

(iii) Uniqueness:

Let $U, V: \mathbb{R}\to\mathcal{L}(\mathcal{H})$ SCUGs with generator $A$.

Using (ii) and „s-lim $A_n B_n = AB$":

$U(t)V(-t)\varphi$ is differentiable for $\varphi\in D(A)$,

and $i\frac{d}{dt}U(t)V(-t)\varphi = U(t)A\,V(-t)\varphi + U(t)\,(-A)V(-t)\varphi = 0$.

$\Rightarrow U(t)V(-t)\varphi = \varphi \quad \forall\varphi\in D(A). \quad \underset{D(A)\text{ is dense}}{\Longrightarrow} \quad U(t) = V(t)$.

(iv) Resolvent identity: Let $\varphi\in D(A)$, $z\in\mathbb{C}_+$.

Differentiate: $i\frac{d}{dt}e^{izt}U(t)\varphi = (A-z)\,e^{izt}U(t)\varphi$.

Thus: $\int_0^R e^{izt}U(t)\varphi\,dt = \int_0^R (A-z)^{-1}\, i\frac{d}{dt}e^{izt}U(t)\varphi\,dt$

bounded op. can be pulled out of integral

$= (A-z)^{-1}\int_0^R i\frac{d}{dt}e^{izt}U(t)\varphi\,dt = (A-z)^{-1} i\big(\underbrace{e^{izR}U(R)\varphi}_{\longrightarrow 0\ (R\to\infty)} - \varphi\big)$.

$= -i\,(A-z)^{-1}\varphi$.

For $\varphi\notin D(A)$: approximate $\varphi_n\to\varphi$, $\varphi_n\in D(A)$.

We can take the limit on the r.h.s. because $(A-z)^{-1}$ is bounded.

We take the limit on the l.h.s.:

$\left\|\int_0^\infty e^{izt}U(t)\varphi_n\,dt - \int_0^\infty e^{izt}U(t)\varphi\,dt\right\| \le \int_0^\infty e^{-\operatorname{Im} z\, t}\|\varphi_n-\varphi\|\,dt \le \text{const.}\,\underbrace{\|\varphi_n-\varphi\|}_{\longrightarrow 0}$. ∎