EXAMPLE (CONTINUED):

3. Let $X = C([0,1])$, $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

$A_k : D_k \subset X \longrightarrow X$, $(A_k f)(x) = \frac{d}{dx} f(x)$, $k=1,2,3,4$.

$D_1 := C^1([0,1])$

$D_2 := \{ f \in D_1 : f(0) = 0 \}$

$D_3 := \{ f \in D_1 : f(0) = f(1) \}$ (periodic bound. cond.)

$D_4 := \{ f \in D_1 : f(0) = 0 = f(1) \}$ (Dirichlet b.c.)

All A_k are closed.

$$\sigma(A_1) = \sigma_p(A_1) = \mathbb{C}$$
$$\sigma(A_2) = \sigma_p(A_2) = \emptyset$$
$$\sigma(A_3) = \sigma_p(A_3) = 2\pi i\mathbb{Z}$$
$$\sigma(A_4) = \mathbb{C}, \quad \sigma_p(A_4) = \emptyset$$

PROOF: • Closedness: Let $(\varphi_n) \subset D_1$, $\varphi_n \to \varphi$ and $A_1\varphi_n = \varphi_n' \longrightarrow \psi$ in X.

This means: $\varphi_n \to \varphi$ uniformly, $\varphi_n' \to \psi$ uniformly.

As we know from real analysis:

$\Rightarrow \varphi \in C^1([0,1]) = D_1$ and $\varphi' = \psi$. $\Rightarrow A_1$ is closed.

Furthermore: if $\varphi_n \in D_k$ ($k = 2,3,4$) also $\varphi \in D_k$

$\Rightarrow A_2, A_3, A_4$ also closed.

• $\sigma(A_1) = \sigma_p(A_1) = \mathbb{C}$: $x \mapsto e^{\lambda x}$ are eigenfct. in D_1 for any $\lambda \in \mathbb{C}$.

• $\sigma(A_2) = \emptyset$: We construct the resolvent explicitly:

Solve $(A_2 - \lambda) f = g$, $g \in X$, for $f \in D_2$:

Find $f \in C^1(I)$ with $f(0) = 0$ s.th. $f' - \lambda f = g$.

Unique solution of the ODE: $f(x) = \int_0^x e^{\lambda(x-t)} g(t)\,dt =: (Sg)(x)$.

We have: S is bounded, $SX \subset D_2$ and $(A_2 - \lambda) Sg = g$.

To show: $S(A_2 - \lambda) g = g$ for $g \in D_2$.

$$S(A_2 - \lambda) g(x) = \int_0^x e^{\lambda(x-t)} (g'(t) - \lambda g(t))\,dt$$
$$= g(x). \quad \Longrightarrow S = R_\lambda(A_2).$$

• $\sigma(A_3)$, $\sigma(A_4)$: exercise. ∎

PROP. 1.1: Let $A : D \subset X \to X$ a linear operator, $\lambda, \mu \in \rho(A)$.
Then:

(i) $R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A)$

(ii) $R_\lambda(A) R_\mu(A) = R_\mu(A) R_\lambda(A)$

(iii) $R_\lambda(A) A \subset A R_\lambda(A)$

(iv) $A R_\lambda(A) = \mathbb{1} + \lambda R_\lambda(A)$.

PROOF: (i)
$$(A-\lambda)^{-1} - (A-\mu)^{-1} = (A-\lambda)^{-1} \left[(A-\mu) - (A-\lambda) \right] (A-\mu)^{-1}$$
$$= (\lambda - \mu) (A-\lambda)^{-1} (A-\mu)^{-1}.$$

(ii) from (i).

(iii), (iv):
$$\forall x \in D : \quad R_\lambda(A) A x = R_\lambda(A)(A-\lambda)x + \lambda R_\lambda(A) x = x + \lambda R_\lambda(A) x.$$

$$\forall x \in X : \quad A R_\lambda(A) x = (A-\lambda) R_\lambda(A) x + \lambda R_\lambda(A) x = x + \lambda R_\lambda(A) x.$$

$\blacksquare$

DEF: Let $U \subset \mathbb{C}$ open. A function $L: U \subset \mathbb{C} \longrightarrow \mathcal{L}(X)$, $z \longmapsto L(z)$, is called <u>analytic</u> if the complex-valued function

$$z \longmapsto f(L(z)x) \in \mathbb{C}$$

is analytic $\forall x \in X$, $\forall f \in X^*$.

dual space of X, bounded linear maps $X \longrightarrow \mathbb{C}$.

[Recall: a function $f: \mathbb{C} \to \mathbb{C}$ is analytic if every point has a neighbourhood where f is represented by f's Taylor series

$$\Updownarrow$$

f is complex differentiable (holomorphic).]

The next lemma is fundamental for most of what we do next.

In fact, for now it is the only method we have for constructing inverse operators, so it is our main tool for constructing resolvents.

LEMMA 1.2: (Neumann series, simplified)
Let X a Banach space, $T \in \mathcal{L}(X)$, with $\|T\| < 1$.
Then $\mathbb{1} - T$ is bijective and $(\mathbb{1} - T)^{-1}$ is also bounded.
Explicitly: $(\mathbb{1}-T)^{-1} = \sum_{n=0}^{\infty} T^n$ (geometric series).

PROOF: Let $S_k = \sum_{n=0}^{k} T^n$.

Then
$$\|S_\ell - S_k\| = \Big\|\sum_{n=k+1}^{\ell} T^n\Big\| \leq \sum_{n=k+1}^{\ell} \|T\|^n \leq \sum_{n=k+1}^{\infty} \|T\|^n \longrightarrow 0 \quad (k \to \infty).$$

So S_ℓ is Cauchy.
Since $\mathcal{L}(X)$ is a Banach space: $\Rightarrow S = \lim_{\ell \to \infty} S_\ell$ exists.

We still have to prove that S is actually the inverse.
This is the same calc. as for the ordinary geom. series:

$$(\mathbb{1}-T)S \xleftarrow{(k\to\infty)} (\mathbb{1}-T)S_k = (\mathbb{1}-T)\sum_{n=0}^{k} T^n = \sum_{n=0}^{k} T^n - \sum_{n=0}^{k} T^{n+1}$$

because T is bounded

$$= \mathbb{1} - T^{k+1} \longrightarrow \mathbb{1}$$ because $\|T^{k+1}\| \leq \|T\|^{k+1} \to 0$.

In the same way we show $S(\mathbb{1}-T) = \mathbb{1}$. ∎

THEOREM 1.3: Let $A : D \subset X \to X$ a linear operator. Then:

(i) $\varrho(A)$ is open, $\sigma(A)$ closed, and the resolvent $z \longmapsto (A-z)^{-1}$ is analytic on $\varrho(A)$.

$B_R(z)$ = ball of radius R around z

For $z_0 \in \varrho(A)$ also $B_{\|R_{z_0}(A)\|^{-1}}(z_0) \subset \varrho(A)$;

and for all z in this ball:

$$R_z(A) = \sum_{n=0}^{\infty} R_{z_0}(A)^{n+1} (z-z_0)^n.$$

(ii) $$\| R_{z_0}(A) \| \geq \frac{1}{\operatorname{dist}(z_0, \sigma(A))}.$$

PROOF:

(i) Let $z_0 \in \rho(A)$. $\forall z \in \mathbb{C}$:

(*) $$A - z = A - z_0 - (z - z_0) = \underbrace{[\mathbb{1} - (z-z_0) R_{z_0}(A)]}_{\text{Need to show: this has bounded inverse.}} \underbrace{(A - z_0)}_{\text{has a bounded inverse because } z_0 \in \rho(A)}.$$

We use the Neumann series to do so:

(**) If $\|(z-z_0) R_{z_0}(A)\| < 1$, then

$1 - (z-z_0) R_{z_0}(A)$ has inverse $\sum_{n=0}^{\infty} R_{z_0}(A)^n (z-z_0)^n$.

$$\Rightarrow z \in \rho(A) \text{ and } R_z(A) = R_{z_0}(A) \sum_{n=0}^{\infty} R_{z_0}(A)^n (z-z_0)^n$$

(in particular, $z \mapsto R_z(A)$ analytic).

(ii) If $z \in$ spectrum: from (*) and (**) it is clear that $(z-z_0) R_{z_0}(A)$ cannot be bounded by one:

So $\forall z \in \sigma(A)$: $|z - z_0| \cdot \|R_{z_0}(A)\| \geq 1$.

$$\Rightarrow \|R_{z_0}(A)\| \geq \frac{1}{\inf_{z \in \sigma(A)} |z - z_0|} = \frac{1}{\operatorname{dist}(z_0, \sigma(A))}.$$

∎

THM 1.4: (Liouville) If a function is analytic on all of $\mathbb{C}$ and bounded, then it is constant.

[PROOF: for $\mathbb{C}$-valued fct.: e.g. Rudin – Real & Complex Analysis.
for op.-valued fct.: use $\mathbb{C}$-valued version, our above def. of analytic, and uniform boundedness principle. ∎]

COROLLARY 1.5: Let $A \in \mathcal{L}(X)$. Then

(i) $\sigma(A) \subset \{z \in \mathbb{C} : |z| \leq \|A\|\}$.

(ii) $\sigma(A) \neq \emptyset$.

PROOF: (i) Let $|z| > \|A\|$. Write

$$(z - A) = \underset{\substack{\text{has}\\ \text{inverse}}}{z}\,(\underbrace{\mathbb{1} - z^{-1}A}_{\text{has inverse because } \|z^{-1}A\| < 1}) : X \to X.$$

$\Rightarrow z \in \varrho(A)$.

(ii) The inverse is $R_z(A) = \sum_{n=0}^{\infty} (z^{-1}A)^n z^{-1}$. (Neumann series) $\quad (**)$

Now we show by contradiction that $\sigma(A)$ can not be empty:

Assume $\sigma(A) = \emptyset$.

Then: $z \mapsto (z-A)^{-1}$ is analytic on all of $\mathbb{C}$, and bounded

because $\|(z-A)^{-1}\| \overset{(**)}{\leq} \frac{1}{|z|} \sum_{n=0}^{\infty} \|z^{-1}A\|^n \overset{\text{conv. geom. series}}{=} \frac{1}{|z| - \|A\|} \xrightarrow[(|z| \to \infty)]{} 0 \quad \binom{*}{**}$.

So $(z-A)^{-1}$ is decaying at ∞, and continuous (even analytic),

so $z \longmapsto \|(z-A)^{-1}\|$ is bounded on $\mathbb{C}$.

$\Rightarrow$ By Liouville: $R_z(A) = \text{const}$.

Because $\binom{*}{**} \longrightarrow 0 \;(|z| \to \infty)$: $R_z(A) = 0 \;\forall z$.

But at the same time: $(A - z) R_z(A) = 1$ (because we assumed that all z are in resolvent set)

Contradiction ↯. ∎

II SYMMETRIC & SELF-ADJOINT OPERATORS

Let $\mathcal{H}$ be a Hilbert space.

DEF: Let $A: D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ linear and densely defined ($\overline{D(A)} = \mathcal{H}$).

The <u>adjoint operator $A^*: D(A^*) \subset \mathcal{H} \longrightarrow \mathcal{H}$</u> is defined as follows:

If for $\varphi \in \mathcal{H}$ there exists $\varphi^* \in \mathcal{H}$ such that

$$\langle \varphi, A\eta \rangle = \langle \varphi^*, \eta \rangle \quad \forall \eta \in D(A),$$

then we say $\varphi \in D(A^*)$ and set $A^*\varphi := \varphi^*$.

CHECK: φ^* is unique, and $\varphi \mapsto \varphi^* = A^*\varphi$ is linear.

RMK: • $A \subset B \Rightarrow A^* \supset B^*$.

• It can happen that $D(A^*) = \{0\}$.

• "$A: D \subset \mathcal{H} \longrightarrow \mathcal{H}$" means always a linear densely defined operator domain D (unless something else is explicitly specified).

PROP. 2.1: For any $A: D \subset \mathcal{H} \to \mathcal{H}$, A^* is closed.

PROOF: Consider a convergent sequence in Γ_{A^*}. Have to show that also the limit is in Γ_{A^*}.

Let $\varphi_n \in D(A^*)$, $\varphi_n \longrightarrow \varphi$ and $A^*\varphi_n \longrightarrow \psi$.

$\forall y \in D(A)$: $\underline{\langle \varphi, Ay \rangle} = \lim_{n\to\infty} \langle \varphi_n, Ay \rangle$ because scalar products are always continuous in their arguments

$\varphi_n \in D(A^*)$ $\quad = \lim_{n\to\infty} \langle A^*\varphi_n, y \rangle$

$= \underline{\langle \psi, y \rangle}.$

Thus: $\varphi \in D(A^*)$ and $A^*\varphi = \psi$.

So $(\varphi, \psi) \in \Gamma_{A^*}$. $\blacksquare$

DEF: $x, y \in \mathcal{H}$ are <u>orthogonal</u> if $\langle x, y \rangle = 0$. Notation: $x \perp y$.

Let $M \subset \mathcal{H}$ a set. <u>Orthogonal complement:</u> $M^\perp := \{x \in V : x \perp y \ \forall y \in M\}$.

RMK:
- $M^\perp$ is a closed subspace.
- $M \subset (M^\perp)^\perp$. If M is a subspace: $(M^\perp)^\perp = \overline{M}$.
- $(\overline{M})^\perp = M^\perp$.

DEF: Let $A : D \subset \mathcal{H} \to \mathcal{H}$.
<u>Kernel of A:</u> $\quad \ker(A) := \{\varphi \in D : A\varphi = 0\}$.

PROP. 2.2: Let $A : D \subset \mathcal{H} \to \mathcal{H}$ densely defined.

Then:
$$\ker(A^*) = (\operatorname{ran} A)^\perp.$$

In particular:
- $(\operatorname{ran} A)^\perp \subset D(A^*)$
- $\ker A^* = \{0\} \Rightarrow \overline{\operatorname{ran} A} = \mathcal{H}.$

PROOF:
- $$\varphi \in \ker A^* \overset{\text{by density}}{\iff} \langle A^*\varphi, \psi\rangle = 0 \ \forall \psi \in D(A)$$
$$\iff \langle \varphi, A\psi\rangle = 0 \ \forall \psi \in D(A)$$
$$\iff \varphi \in (\operatorname{ran} A)^\perp.$$

- $(\operatorname{ran} A)^\perp \subset D(A^*)$ clear because $\ker(A^*) \subset D(A^*)$ by def.
- For $\{0\} = \ker A^*$:
$$\mathcal{H} = \{0\}^\perp = (\ker A^*)^\perp = (\operatorname{ran} A^\perp)^\perp = \overline{\operatorname{ran} A}. \qquad \blacksquare$$

DEF: Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be densely defined.

<u>A is symmetric</u> if $A \subset A^*$, i.e. $\langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle \ \forall \varphi, \psi \in D(A)$.

<u>A is selfadjoint</u> if $A = A^*$ (i.e. symmetric <u>and</u> $D(A^*) = D(A)$).

RMK: For bounded operator: symmetric = selfadjoint = hermitian.

THM: (Hellinger – Töplitz)

Let $\mathcal{H}$ a Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ linear and symmetric, i.e. $\langle y, Ax\rangle = \langle Ay, x\rangle \ \forall x, y \in \mathcal{H}$.

Then A is bounded.

PROOF: on Friday.

uses Closed-Graph Theorem ... .