ADVANCED MATHEMATICAL PHYSICS (with Jérémy Sok)

TOPICS "QM ∩ Functional Analysis"

- selfadjoint operators, spectrum
- time-dep. Schrödinger equation
- symmetries
- scattering theory
- essential vs. discrete spectrum
- decay of the ground state

⋮

MODALITIES

hit F5 to check for updates!

www.nielsbenedikter.de ⟶ Teaching : USER: mathphys
PW: yukawa

- Mon, Tue: lectures, Fri: Seminar.
- assignements are due every 2nd Friday before seminar
- requirements:
  - approx. 50% averaged
    + serious attempt at every assignment
  - seminar talk of 40 min + 5 min discussion
    + written summary (with list of references – don't plagiarize!)
    within 3 days after the talk.
  - participation in the seminar

LITERATURE: these lecture notes are the main reference!

To read up more: mostly Teschl's book.

QUANTUM MECHANICS [Teschl 2.1 / phys. books]

A qm. theory consists of a Hilbert space $(\mathcal{H}, \langle , \rangle)$ and some self-adjoint operators $A_1, A_2 \dots$ and H.

- Vectors $\psi \in \mathcal{H}$ describe states of the system.
- $A_1, A_2 \dots$ correspond to observables (e.g. position, momentum...)
- H corresponds to the observable "total energy"
- States ψ evolve in time with the SE

initial value problem:
$$\begin{cases} i \frac{d}{dt} \psi(t) = H\psi(t), \\ \psi(0) = \psi_0 \in \mathcal{H}. \end{cases}$$

- In experiments we get the averages $\langle \psi, A_i \psi \rangle \in \mathbb{R}$.

Concrete Example:

$\mathcal{H} = L^2(\mathbb{R}^3)$, $H = -\Delta$ (kinetic energy) or $H = -\Delta + V$ (Schrödinger operator)

$A_1 = p = -i\nabla$, $A_2 = x$, $A_3 =$ angular momentum.

p: momentum, x: position

Interpretation of the wave function $\psi \in L^2(\mathbb{R}^3)$:

For $\Omega \subset \mathbb{R}^3$: $\int_\Omega |\psi(x)|^2 \, dx =$ probability to find a particle in Ω.

For this reason states are typically normalized:
$$\|\psi\|_{L^2}^2 = \int |\psi(x)|^2 \, dx = 1.$$

0 BASIC FUNCTIONAL ANALYSIS [Teschl 0.2–0.6]

DEF: <u>Banach space:</u> a $\mathbb{C}$-vector space with norm $\|\cdot\|$ that is complete, i.e. in which every Cauchy sequence converges:

$$\Big(\underbrace{\forall \varepsilon>0\ \exists N\in\mathbb{N}:\ k,\ell\geq N \Rightarrow \|x_k-x_\ell\|<\varepsilon}_{\text{Def. of Cauchy sequence}}\Big) \Longrightarrow (x_k) \text{ converges } (k\to\infty).$$

EXAMPLE: <u>L^p-spaces:</u>

$$L^p(\mathbb{R}^d) := \{ [f] : f:\mathbb{R}^d\to\mathbb{C} \text{ measurable and } \int |f|^p\,dx<\infty \}$$

$$\|[f]\|_p := \left(\int |f|^p dx\right)^{1/p}$$

equiv. classes of fcts. which are the same a.e. (except for sets of zero measure)

and

$$L^\infty(\mathbb{R}^d) := \{ [f] : f:\mathbb{R}^d\to\mathbb{C} \text{ measurable and } \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |g(x)|<\infty \}$$

$$\|[f]\|_\infty := \inf_{g\in[f]} \sup_{x\in\mathbb{R}^d} |g(x)|.$$

are Banach spaces. In the future we don't write $[f]$, only f.

DEF: Let X, Y vector spaces with norm.

An <u>operator</u> is a linear map $A: X\to Y$, i.e.

$$A(x+x') = Ax + Ax' \quad \text{and} \quad A(\lambda x) = \lambda Ax \qquad \forall x,x'\in X, \lambda\in\mathbb{C}.$$

A is <u>bounded</u> if

$$\underbrace{\sup_{x\in X, \|x\|=1} \|Ax\|}_{=:\ \|A\|\ \text{operator norm}} < \infty.$$

LEMMA: A is continuous if and only if A is bounded.

PROOF: "$\Rightarrow$": A continuous $\Rightarrow$ A is continuous in zero

$\Rightarrow \exists \delta > 0: \|x\| < \delta \Rightarrow \|Ax\| < 1.$

So for $x \neq 0$: $\|Ax\| = \|A \frac{x}{\|x\|} \frac{\delta}{2}\| \frac{2}{\delta} \|x\|$

$\leq \frac{2}{\delta} \|x\|$

$\Rightarrow$ A is bounded.

"$\Leftarrow$": Let $c = \sup_{\|x\|=1} \|Ax\|$. A is bounded means: $c < \infty$.

Then for $x \neq y$:

$\|A(x-y)\| = \|A \frac{x-y}{\|x-y\|}\| \|x-y\| \leq c \|x-y\|.$

$\Rightarrow$ A is continuous (in fact even Lipschitz continuous).

$\blacksquare$

DEF: $\mathcal{L}(X,Y) := \{$ bounded operators $X \longrightarrow Y\}$

$\mathcal{L}(X) := \mathcal{L}(X,X)$

$\mathcal{L}(X, \mathbb{C}) =: X^*$ <u>dual space</u>

RMK:
- $\mathcal{L}(X,Y)$ is a vector space with addition $(A+B)x = Ax + Bx$ and scalar multiplication $(\lambda A)x := \lambda(Ax)$.
- If Y is a Banach space, so is $\mathcal{L}(X,Y)$ with the operator norm $\|A\|$.
- $\|Ax\| \leq \|A\| \|x\|$;
  for $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$: $\|BA\| \leq \|B\| \|A\|$.

DEF: <u>Hilbert space:</u>

A complex vector space $\mathcal{H}$ with a scalar product $\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$, i.e.

- $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ and $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle$ $\forall x, y, z \in \mathcal{H}, \lambda \in \mathbb{C}$. (Linearity)
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Physicist convention: antilin. in <u>first</u> variable, lin. in 2nd. Mathematicians usually do it the other way around.)
- $\langle x, x \rangle > 0 \quad \forall x \neq 0$,

that is a Banach space w.r.t. the induced norm $\|x\| := \sqrt{\langle x, x \rangle}$.

RMK: Cauchy–Schwarz inequality: $\langle x, y \rangle \leq \|x\| \, \|y\|$.

EX: $L^2(\mathbb{R}^d)$ with $\langle f, g \rangle = \int \overline{f(x)}\, g(x) \, dx$.

THM: (FRÉCHET–RIESZ) Let $f \in \mathcal{H}^*$. Then there exits one and only one $x \in \mathcal{H}$ such that:

$$f(y) = \langle x, y \rangle \quad \forall y \in \mathcal{H}.$$

$$\text{„} f = \langle x, \cdot \rangle = \langle x | \text{"}$$

„Dirac Bra-Ket notation"

THM: If X, Y are Banach spaces and $A: X \to Y$ is bounded and 1-to-1, then $A^{-1}: Y \to X$ is bounded, too.

DEF: Let X, Y be vector spaces. Then $X \times Y$ with
$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2), \quad \lambda(x,y) = (\lambda x, \lambda y)$$
is a vector space: the <u>direct sum</u> $X \oplus Y$.

If X, Y have a norm: $\|(x,y)\|^2 := \|x\|^2 + \|y\|^2$.

If $A : X \to Y$ is linear: $\Gamma_A := \{(x,y) \in X \oplus Y : y = Ax\}$ is the <u>graph of A</u>.

A is <u>closed</u> if Γ_A is closed in $X \oplus Y$.

(i.e. if you have a sequence in Γ_A that converges, then also the limit is in Γ_A.)

THM: (Closed graph thm) Let X, Y Banach spaces, $A : X \to Y$ linear. Then:
A bounded $\iff$ A closed. (defined on all the space!)

PROOF: "$\Rightarrow$" Let A be continuous. Let $(x,y) \in X \oplus Y$ and (x_n, y_n) a sequence in Γ_A, with $x_n \to x$, $y_n \to y$.

Then $y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} A x_n = Ax$.

So $(x,y) \in \Gamma_A$. So Γ_A is closed.

"$\Leftarrow$": Let $\Gamma_A \subset X \oplus Y$ closed.
Then Γ_A by itself is a Banach space.

Then $$P : \Gamma_A \longrightarrow X$$
$$(x,y) \longmapsto x$$

is linear, bijective and continuous.

By previous Thm.: $P^{-1} : X \longrightarrow \Gamma_A$ is also bounded.

$$\text{So} \quad \|Ax\| \leq \sqrt{\|x\|^2 + \|Ax\|^2} = \|(x, Ax)\|$$
$$= \|P^{-1}x\| \leq \|P^{-1}\| \, \|x\|.$$

So A is continuous. ■

However: In physics we encounter unbounded (but closed) operators!
This is only possible because they are not defined on all X, rather only on a dense subspace.

$\longrightarrow$ next chapter.

(Notice also: a <u>bounded</u> op. def. on a dense subspace can always by continuity be extended to a bounded op. on <u>all</u> space.)

I SPECTRUM & RESOLVENT [Teschl 2.4]

DEF: Let X a Banach space, $D \subset X$ a linear subspace.

<u>Linear operator in X with domain D:</u>
a linear map $A : D \longrightarrow X$,
i.e.
$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A x_1 + \lambda_2 A x_2 \qquad \begin{array}{l}\forall \lambda_1, \lambda_2 \in \mathbb{C} \\ \forall x_1, x_2 \in D\end{array}$$

[An operator consists of a "mapping prescription" <u>and</u> a domain!]

<u>range of A:</u> $\operatorname{ran}(A) := \{ Ax : x \in D \}$.

A is <u>densely defined</u> if $\overline{D} = X$.

A is <u>closed</u> if its graph $\Gamma_A := \{ (x,y) : x \in D,\ y = Ax \}$ is closed in $X \oplus X$.

An operator $B : D(B) \subset X \longrightarrow X$ is called <u>extension</u> of A if $D(A) \subset D(B)$ and $Ax = Bx \quad \forall x \in D(A)$.
We write: $A \subset B$.

[RMKS:

- $A = B$ iff $D(A) = D(B)$ and $Ax = Bx \ \forall x \in D(A)$.
- We often don't write the identity: $A - \lambda = A - \lambda \mathbb{1} \quad \forall \lambda \in \mathbb{C}$.
  (In general it is tricky to define the sum of two unbounded operators — you could have $D(A) \cap D(B) = \emptyset$. Luckily the domain of the identity operator $\mathbb{1}$ is all the space, so $A - \lambda$ is naturally well-defined on $D(A)$.)
]

<u>resolvent set:</u> $\rho(A) = \{ \lambda \in \mathbb{C} : (A - \lambda) : D(A) \to X$ is 1-to-1 and $(A-\lambda)^{-1}$ is bounded $\}$.

(Again: $(A-\lambda)^{-1}$ bounded means $\|(A-\lambda)^{-1}\| = \sup_{\substack{x\in X \\ \|x\|=1}} \|Ax\| < \infty$.)

resolvent of A: the function $\lambda \mapsto R_\lambda(A) := (A-\lambda)^{-1}$.

spectrum of A: $\sigma(A) = \mathrm{spec}(A) := \mathbb{C} \setminus \varrho(A)$.

point spectrum: $\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}$

WARNING: $\sigma_p(A) \subset \sigma(A)$, but in general: $\sigma_p(A) \neq \sigma(A)$.

EXAMPLES:

1. $X = \mathbb{C}^n$, $A \in \mathcal{L}(X)$.
   Then: $\sigma(A) = \sigma_p(A) = \{\text{eigenvalues of } A \text{ as a matrix}\}$.

2. $X = L^2(\mathbb{R})$, $D = \{f \in L^2(\mathbb{R}) : \int |x f(x)|^2 \, dx < \infty\}$,
   $Af(x) := x f(x)$. "position operator"
   Then: $\sigma(A) = \mathbb{R}$, $\sigma_p(A) = \emptyset$, $\varrho(A) = \mathbb{C} \setminus \mathbb{R}$.

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plan for Tuesday:
- more examples
- Neumann series
- self-adjointness
- resolvent
- ...