COMPACT OPERATORS

DEFINITION: Let X and Y be Banach spaces.
An operator $T \in \mathcal{L}(X,Y)$ is called compact (or completely continuous) if T takes bounded sets in X into precompact sets in Y.
Equivalently, T is compact if and only if for every bounded sequence $(x_n)_{n\geq 1} \subset X$, $(Tx_n)_{n\geq 1}$ has a subsequence convergent in Y.

EXAMPLE: (finite rank operators)
We say that $T \in \mathcal{L}(X,Y)$ is a finite rank operator if $\dim(\operatorname{ran}(T)) < \infty$.
Hence if $(x_n)_{n\geq 1} \in (X)_1$, then $(Tx_n)_{n\geq 1}$ is also a bounded sequence (because $\|T\| < \infty$) in the finite-dimensional space $\operatorname{ran}(T) \cong \mathbb{C}^N$.
So, we can extract a convergent subsequence by classical Bolzano-Weierstrass Theorem.
(all the norms on $\operatorname{ran}(T) \cong \mathbb{C}^N$ are equivalent, so it suffices to identify $\operatorname{ran}(T)$ with $\mathbb{C}^N$ as a vector space and the induced topology must be the usual topology on $\mathbb{C}^N$.)

THEOREM (I) (weak into norm convergence) [THM VI.11 Reed Simon]
A compact operator maps weakly convergent sequences into norm convergent sequences.

PROOF: Suppose $x_n \xrightarrow{w} x \Leftrightarrow$
$\Leftrightarrow f(x_n) \longrightarrow f(x)$, for all $f \in X^*$

Step ①: Define $\hat{x}_n \in X^{**}$ by $\hat{x}_n(f) = f(x_n)$, for all $f \in X^*$

By fixing $f \in X^*$, we get that $(f(x_n))_{n \geq 1}$ is a strongly convergent sequence and, in particular, it is bounded, ie $\sup_{n \in \mathbb{N}} |f(x_n)| < \infty$

By uniform boundedness theorem, we get that
if $\sup_{n \geq 1} |\hat{x}_n(f)| < \infty$, then

$$\sup_{n \geq 1} \|\hat{x}_n\|_{X^{**}} < \infty$$

Hence $\sup_{n \geq 1} \|x_n\| = \sup_{n \geq 1} \|\hat{x}_n\|_{X^{**}} < \infty$

(using Hahn-Banach theorem: if $x \in X$, then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**})

Hence $\|x_n\|$ are bounded.

Step ② Let $y_n = Tx_n$ and $y = Tx$, then

$$\ell(y_n) - \ell(y) = (\ell T)(x_n - x) =$$
$$= (T'\ell)(x_n - x) \quad \text{for any } \ell \in Y^*$$

Thus y_n converges weakly to $y = Tx$ in Y

Step 3: Assume by contradiction that $y_n \not\to y$ in the norm topology

Then there is an $\varepsilon > 0$ and a subsequence $(y_{n_k})_{k\geq 1}$ of $(y_n)_{n\geq 1}$ so that $\|y_{n_k} - y\| \geq \varepsilon$

Since the sequence $(x_{n_k})_{k\geq 1}$ is bounded and T is compact, $(y_{n_k})_{k\geq 1}$ has a subsequence

that converges to a $\tilde{y} \neq y$. This subsequence must then also converge weakly to $\tilde{y}$.

But, this is impossible since y_n converges weakly to y.

Thus: $y_n \xrightarrow{\|\cdot\|} y$

$\blacksquare$

THEOREM (II) [THM VI.12, Reed Simon]
Let X, Y, Z be Banach spaces and
$T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$

(a) If $(T_n)_{n\geq 1}$ are compact and $T_n \to T$ in the norm topology, then T is compact.

(c) If T or S is compact, then ST is compact.

PROOF: (a) Let $(x_m)_{m\geq 1}$ be a sequence in the unit ball of X.

Since T_n is compact for each $n \geq 1$, we can find a subsequence of $(x_m)_{m\geq 1}$, call it $(x_{m_k})_{k\geq 1}$ such that $T_n x_{m_k} \longrightarrow y_n$ for $n \geq 1$. as $k \to \infty$ $(*)$.

We want to show that $(T_n x_{m_k})_{k\geq 1}$ is convergent in norm for all $n \in \mathbb{N}$. We will construct by induction a sequence $((\tilde{x}_{m,n})_{m\geq 1})_{n\geq 1}$ of subsequences of $(x_m)_{m\geq 1}$ such that for all $n \in \mathbb{N}$ this sequence has the following properties:

(1) $(\tilde{x}_{m,n+1})_{m\geq 1}$ is a subsequence of $(\tilde{x}_{m,n})_{m\geq 1}$

(2) for all $n_1 \in \mathbb{N}$ such that $1 \leq n_1 \leq n$ we have that $(T_{n_1} \tilde{x}_{m,n})_{m\geq 1}$ is norm-convergent.

The induction: step 1 for $n=1$, we pick a subsequence $(x_{m_k})_{k\geq 1}$ such that $(T_1 x_{m_k})_{k\geq 1}$ is convergent (by $(*)$, for $n=1$)

step 2. for $n = n_0 \geq 1$, assume that there exists a sequence $(\tilde{x}_{m,n_0})_{m\geq 1}$ satisfying the properties (1) and (2).

<u>for $n = n_0 + 1$</u>

since T_{n_0+1} is compact, there exists a subsequence $(\tilde{x}_{m_k, n_0})$ such that $(T_{n_0+1} \tilde{x}_{m_k, n_0})_{k \geq 1}$ is convergent. Hence, $(\tilde{x}_{m, n_0+1})_{m \geq 1}$ also satisfies the properties (1) and (2) for $n = n_0 + 1$.

Thus, such a sequence $((\tilde{x}_{m,n})_{m \geq 1})_{n \geq 1}$ exists.

Consider $(x_{m_k})_{k \geq 1} = (\tilde{x}_{k,k})_{k \geq 1} \subset (x_m)_{m \geq 1}$, then

For fixed $n_0 \in \mathbb{N}$, and for $n_0 \leq k$, by (1) we have that:

$$(x_{m_k}) = (\tilde{x}_{k,k}) \subset (\tilde{x}_{k, n_0})$$

Hence the sequence $(T_{n_0} \tilde{x}_{k,k})_{k \geq 1}$ converges

Hence for $n_0 \in \mathbb{N}$, $(T_{n_0} x_{m_k})_{k \geq 1}$ converges in $\|\cdot\|_Y$ and $(T_{n_0} x_{m_k}) \xrightarrow{k \to \infty} y_{n_0}$

Now we will show that $(y_n)_{n \geq 1}$ is a Cauchy sequence.

$$\|y_n - y_\ell\| \leq \|y_n - T_n x_{m_k}\| + \|T_n x_{m_k} - T_\ell x_{m_k}\| + \|T_\ell x_{m_k} - y_\ell\|$$

but $\|y_n - T_n x_{m_k}\| \longrightarrow 0$ as $k \to \infty$,

$\|T_n x_{m_k} - T_\ell x_{m_k}\| \leq \|T_n - T_\ell\| \cdot \|x_{m_k}\| \leq \|T_n - T_\ell\| \cdot 1 \longrightarrow 0$ as $n, \ell \to \infty$, since T_n is a Cauchy sequence in operator norm topology

and $\|T_\ell x_{m_k} - y_\ell\| \longrightarrow 0$ as $k \to \infty$

Hence $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that for $n, \ell \geq N_0$ we have that $\|y_n - y_\ell\| < \varepsilon$.

Hence $(y_n)_{n \geq 1}$ is a Cauchy sequence. Thus, there exists a unique $y \in Y$ such that $y_n \longrightarrow y$ in $\|\cdot\|_Y$ as $n \to \infty$.

In a similar way,

$$\begin{aligned}\|Tx_{n_k} - y\| &\le \|Tx_{n_k} - T_n x_{n_k}\| + \|T_n x_{n_k} - y_n\| + \|y_n - y\| \\ &\le \|T - T_n\| \cdot \|x_{n_k}\| + \|T_n x_{n_k} - y_n\| + \|y_n - y\| \\ &\le \frac{\varepsilon}{3} \cdot 1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for every $\varepsilon > 0$ and large enough $k \in \mathbb{N}$

Hence $Tx_{n_k} \longrightarrow y$ as $k \to \infty$ in the norm topology.

Hence T is compact

$\blacksquare$

(c). Assume that S is compact and $(x_n)_{n\geq 1}$ is a bounded sequence in X

we will show that $(STx_n)_{n\geq 1}$ has a convergent subsequence.

$\|Tx_n\|_Y \le c \cdot \|x_n\|_X < \infty$, since $T \in \mathcal{L}(X,Y)$

Hence $(Tx_n)_{n\geq 1}$ is also bounded.

Since S is compact there is a subsequence of $(S(Tx_n))_{n\geq 1}$, call it $(S(Tx_{n_k}))_{k\geq 1}$, that converges.

Hence, we found a subsequence $(STx_{n_k})_{k\geq 1}$ of $(STx_n)_{n\geq 1}$ that converges.

Thus, ST is compact.

Now, assume that T is compact.
and $(x_n)_{n\geq 1}$ is a bounded sequence in X.

then there exists a subsequence of $(Tx_n)_{n\geq 1}$, call it $(Tx_{n_k})_{k\geq 1}$ that converges, e.g. $\exists y \in Y$ such that $Tx_{n_k} \xrightarrow{k\to\infty} y$ in the norm topology.

Hence $\| STx_{n_k} - Sy \|_Z = \| S(Tx_{n_k} - y) \|_Z \leq$
$$\leq C \cdot \| Tx_{n_k} - y \|_Y$$
$$\leq C \cdot \varepsilon$$
For every $\varepsilon > 0$ and for large enough $k \in \mathbb{N}$

Hence $(STx_{n_k})_{k\geq 1}$ converges and this means that ST is compact.

$\boxtimes$

THEOREM (III) (THM VI.13, Reed Simon)
Let H be a separable Hilbert space. THEN, Every compact operator on H is the norm limit of a sequence of operators of finite rank.

PROOF: Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis in H.

Define $\lambda_n := \sup\{\|T\psi\| : \psi \in (\mathrm{span}(\varphi_1, \dots, \varphi_n))^{\perp}, \|\psi\| = 1\}$

λ_n is clearly monotone decreasing and since $\|T\psi\| \geq 0$, we have that the limit of $(\lambda_n)_{n\geq 1}$ (as $n \to \infty$) exists and $\lim_{n\to\infty} \lambda_n := \lambda \geq 0$

We will show that $\lambda = 0$.

Assume that $\lambda > 0$, then there exists a sequence $(\psi_n)_{n\geq 1}$ such that $\psi_n \in (\mathrm{span}(\varphi_1, \dots, \varphi_n))^{\perp}$, $\|\psi_n\| = 1$ and has $\|T\psi_n\| \geq \lambda/2$

Claim: $\psi_n \xrightarrow{w} 0$, as $n \to \infty$
(ie $\langle \varphi, \psi_n \rangle \longrightarrow 0 \quad \forall \varphi \in H$)

Let $\varphi \in H$, then $\varphi = \varphi_n^1 + \varphi_n^2$,
where $\varphi_n^1 \in \mathrm{span}(\varphi_1, \dots \varphi_n)$ and
$\varphi_n^2 \in (\mathrm{span}(\varphi_1, \dots, \varphi_n))^{\perp}$.
Hence, $\varphi_n^2 \longrightarrow 0$, as $n \to \infty$
and $\langle \psi_n, \varphi_n^1 \rangle = 0$ since $\psi_n \perp \varphi_n^1$

Hence, $|\langle \psi_n, \varphi \rangle| = |\langle \psi_n, \varphi_n^1 + \varphi_n^2 \rangle| =$
$= |\langle \psi_n, \varphi_n^2 \rangle|$
$\overset{C-S}{\leq} \underset{\overset{\shortparallel}{1}}{\|\psi_n\|} \cdot \underbrace{\|\varphi_n^2\|}_{\longrightarrow 0}$

Hence $\varphi_n \longrightarrow 0$ weakly, as $n \longrightarrow \infty$

Then $T\varphi_n \longrightarrow 0$ in $\|\cdot\|_H$ as $n \longrightarrow \infty$
(using theorem (I))

Hence, $0 \leq \lambda/2 \leq \|T\varphi_n\| \xrightarrow{n\to\infty} 0 \Rightarrow$

$\Rightarrow \lambda = 0$

Define $T_n : H \longrightarrow H$ by $T_n = \sum_{j=1}^{n} (\varphi_j, \cdot) T\varphi_j$,
for $n \in \mathbb{N}$

Notice that $(T_n)_{n\geq 1}$ is a sequence of finite rank operators on H.

$$\left\| \sum_{j=1}^{n} (\varphi_j, \cdot) T\varphi_j - T \right\| =$$

$$= \sup_{\substack{\psi \in H \\ \|\psi\|=1}} \left\| \left(\sum_{j=1}^{n} (\varphi_j, \cdot) T\varphi_j - T \right) \psi \right\|_H$$

$$= \sup_{\substack{\psi \in H, \|\psi\|=1 \\ \psi = \psi_1 + \psi_2 \in \mathrm{span}(\varphi_1, \dots, \varphi_n) \oplus (\mathrm{span}(\varphi_1, \dots, \varphi_n))^\perp}} \left\| \left(\sum_{j=1}^{n} (\varphi_j, \psi_1) T\varphi_j - T\psi_1 \right) + \left(\sum_{j=1}^{n} (\varphi_j, \psi_2) T\varphi_j - T\psi_2 \right) \right\|_H$$

$$= \sup_{\substack{\psi \in H, \|\psi\|=1 \\ \psi = \psi_1 + \psi_2 \in \mathrm{span}(\varphi_1, \dots, \varphi_n) \oplus \mathrm{span}(\varphi_{n+1}, \dots)}} \| (T\psi_1 - T\psi_1) + (0 - T\psi_2) \|_H$$

$$= \sup_{\substack{\psi_2 \in (\mathrm{span}(\varphi_1, \dots, \varphi_n))^\perp \\ \|\psi_2\| = 1}} \| T\psi_2 \|_H = \lambda_n, \text{ but } \lambda_n \xrightarrow{n\to\infty} 0$$

Hence, $\sum_{j=1}^{n} (\varphi_j, \cdot) T\varphi_j \longrightarrow T$, as $n \to \infty$

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REFERENCES:

- Reed M., Simon B., "Methods of Modern Mathematical Physics", vol. 1.